ANALYSIS ON REAL LIE GROUPS

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1. Introduction and motivations

This course is about spherical functions on a semisimple real Lie group, and the spherical transform. The later may be viewed as a non-Abelian version of the Fourier transform, where the spherical functions play the role of the complex exponential. In this foreword, we introduce the spherical functions and spherical transform from the point of view of representation theory, through the motivating example of the trace formula.

1.1. **Basic Fourier theory.** The central result in Fourier theory is that under suitable assumptions, a function can be recovered from its Fourier transform, and the inversion formula is particularly simple and elegant.

Theorem 1.1 (Fourier inversion theorem)**.** *Let f be a continuous, integrable function on $\mathbb{R}$. For $\nu \in \mathbb{R}$ define*

$$(\mathscr{F}f)(\nu) = \int_{\mathbb{R}} f(x) \exp(-2i\pi x\nu)\, dx.$$

Assume that $\mathscr{F}f$ is integrable. Then for all $x \in \mathbb{R}$ we have

$$f(x) = \int_{\mathbb{R}} (\mathscr{F}f)(\nu) \exp(2i\pi x\nu)\, d\nu.$$

Depending on one's purpose, the assumptions on f may be relaxed or strengthened in many different ways. A classical result characterises the image of the space of smooth, compactly supported functions f by the Fourier transform. Although in previous theorem ν was a real variable, in this situation one may define $(\mathscr{F}F)(\nu)$ for any complex number ν by the same formula. A holomorphic function h on $\mathbb{C}$ is Paley-Wiener if there exists $R > 0$ such that for all $N \geq 0$ there exists $C_N \in \mathbb{R}$ such that

$$|h(\nu)| \leq C_N(1 + |\nu|)^{-N} e^{R|\Im(\nu)|}$$

for all $\nu \in \mathbb{C}$.

Theorem 1.2 (Paley-Wiener)**.** *The Fourier transform* $\mathscr{F}$ *is a bijection between the space of smooth, compactly supported functions on* $\mathbb{R}$ *and the space of Paley-Wiener functions on* $\mathbb{C}$*.*

These results naturally extend to the Euclidean space $\mathbb{R}^n$. More generally, there is a Fourier inversion formula for locally compact Abelian topological groups (this is the object of Pontryagin duality). However, in the course of history the need arose for non-Abelian generalisations of the Fourier transform. As a motivating example, we briefly discuss the trace formula.

1.2. **The trace formula.** Let G be a locally compact topological group. Assume that G has a non-trivial Radon measure μ that is both left-invariant and right-invariant under the action of G (that is G is unimodular). For instance one may think of $G = \mathbb{R}$ endowed with the usual measure dx. A less trivial example is $G = \mathrm{SL}_2(\mathbb{R})$ endowed with the measure given by $d\mu(g) = \frac{1}{y}dxdyd\theta$ if

$$g = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (x \in \mathbb{R}, y > 0, \theta \in [0, 2\pi)).$$

Let Γ be a discrete subgroup of G. Then the measure μ descends to a right-invariant measure on the quotient space $X = \Gamma\backslash G$, that we denote by $\mu_{\Gamma\backslash G}$. Moreover we may – and we will – identify functions on $\Gamma\backslash G$ with left-Γ-invariant functions on G. To keep things simple, assume that X is compact. Consider the regular right representation R of G of $L^2(X)$, defined for $g \in G$ and $\varphi \in L^2(X)$ by

$$(R(g)\varphi)(x) = \varphi(xg) \qquad \text{for } \mu-\text{almost all } x \in G.$$

Given a continuous compactly supported function $f : G \to \mathbb{C}$ define an operator $R(f)$ on $L^2(X)$ by

$$(R(f)\varphi)(x) = \int_G f(g)\varphi(xg)\,d\mu(g) \qquad \text{for } \mu-\text{almost all } x \in G.$$

Then by left-invariance of μ and by Fubini's theorem we have

$$\begin{aligned}(R(f)\varphi)(x) &= \int_G f(x^{-1}g)\varphi(g)\,d\mu(g) \\ &= \int_{\Gamma\backslash G} \sum_{\delta\in\Gamma} f(x^{-1}\delta g)\varphi(\delta g)\,d\mu_{\Gamma\backslash G}(g) = \int_{\Gamma\backslash G} K_f(x,g)\varphi(g)\,d\mu_{\Gamma\backslash G}(g),\end{aligned}$$

where

$$K_f(x,g) = \sum_{\delta\in\Gamma} f(x^{-1}\delta g).$$

Note that since f has compact support, for any fixed $x, g \in G$ the sum above is finite, and in particular $K_f(x, g)$ is well-defined (and continuous). Thus we have realised the operator $R(f)$ as an integral operator with kernel K_f. Assume moreover that the operator $R(f)$ is trace class, and that its trace is given by

$$\mathrm{tr}(R(f)) = \int_{\Gamma\backslash G} K_f(x, x) d\mu_{\Gamma\backslash G}(x). \tag{1.1}$$

By general theory, this is in particular the case if G is a real Lie group and f is smooth. The trace formula stems from evaluating this trace in two different ways. On the one hand, given $\delta \in \Gamma$, let $\{\delta\}$ be its conjugacy class in Γ and let $\Gamma_\delta = \{\gamma \in \Gamma : \gamma\delta\gamma^{-1} = \delta\}$ be its centraliser in Γ. Then we can rewrite the kernel K_f as

$$K_f(x, g) = \sum_{\{\delta\}} \sum_{\gamma \in \Gamma_\delta\backslash\Gamma} f(x^{-1}\gamma^{-1}\delta\gamma g).$$

Reporting this in (1.1) and assuming we may swap integration and summation order we obtain

$$\begin{aligned}\mathrm{tr}(R(f)) &= \sum_{\{\delta\}} \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma_\delta\backslash\Gamma} f(x^{-1}\gamma^{-1}\delta\gamma x)\, d\mu_{\Gamma\backslash G}(x) \\ &= \sum_{\{\delta\}} \int_{\Gamma_\delta\backslash G} f(y^{-1}\delta y)\, d\mu_{\Gamma_\delta\backslash G}(y).\end{aligned}$$

Now let G_δ be the centraliser of δ in G. Assume that the centralisers G_δ are closed, unimodular subgroups (this is in particular the case if G is a semisimple real Lie group). Then we may decompose the quotient measure $d\mu_{\Gamma_\delta\backslash G}$ as $d\mu_{\Gamma_\delta\backslash G_\delta} d\mu_{G_\delta\backslash G}$ and thus we have

$$\int_{\Gamma_\delta\backslash G} f(y^{-1}\delta y)\, d\mu_{\Gamma_\delta\backslash G}(y) = \mathrm{Vol}(\Gamma_\delta\backslash G_\delta) I_\delta(f)$$

where

$$I_\delta(f) = \int_{G_\delta\backslash G} f(y^{-1}\delta y)\, d\mu_{G_\delta\backslash G}(y).$$

The integral $I_\delta(f)$ is called an orbital integral because as y ranges over $G_\delta\backslash G$, the integrand $y^{-1}\delta y$ ranges precisely over the orbit of δ by conjugation in G. Thus we have obtained the so-called geometric side of the trace formula

$$\mathrm{tr}(R(f)) = \sum_{\{\delta\}} \mathrm{Vol}(\Gamma_\delta\backslash G_\delta) I_\delta(f). \tag{1.2}$$

On the other hand, it can be shown that the representation R decomposes as a Hilbert direct sum of irreducible unitary representations π with finite multiplicity m_π:

$$R \simeq \bigoplus_{\pi \in \widehat{G}} m_\pi \pi,$$

where $\widehat{G}$ is the unitary dual of G, that is, the set of equivalence classes of irreducible unitary representations of G. Given a closed subspace $V \subset L^2(X)$ that is invariant by R, by construction the operator $R(f)$ preserves V. In particular for each $\pi \in \widehat{G}$ we may consider the operator $\pi(f)$ obtained by restricting $R(f)$ to any closed invariant subspace V_π on which the regular right representation of G is isomorphic to π. Thus we obtain the so-called spectral side of the trace formula

$$\operatorname{tr}(R(f)) = \sum_{\pi \in \widehat{G}} m_\pi \operatorname{tr}(\pi(f)). \tag{1.3}$$

Assembling (1.2) and (1.3) we arrive – subject to our running assumptions – at the following identity, known as the trace formula

$$\sum_{\pi \in \widehat{G}} m_\pi \operatorname{tr}(\pi(f)) = \sum_{\{\delta\}} \operatorname{Vol}(\Gamma_\delta \backslash G_\delta) I_\delta(f).$$

Before proceeding any further we give an example that relates the trace formula to Fourier theory. Take $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. Since in this case G is commutative the conjugacy classes are trivial and thus the geometric side is given by

$$\operatorname{tr}(R(f)) = \sum_{\delta \in \mathbb{Z}} f(\delta).$$

On the other hand the irreducible unitary representations π of $G = \mathbb{R}$ are unitary characters $\pi_\nu : x \mapsto \exp(2i\pi\nu x)$ where $\nu \in \mathbb{R}$. A subspace of $L^2(\mathbb{R}/\mathbb{Z})$ that is equivalent to π_ν is spanned by a non-zero function $\varphi_\nu \in L^2(\mathbb{Z}/\mathbb{R})$ satisfying $R(x)\varphi_\nu = \exp(2i\pi\nu x)\varphi_\nu$ for all $x \in \mathbb{R}$, that is

$$\varphi_\nu(x+y) = \exp(2i\pi\nu x)\varphi_\nu(y) \tag{1.4}$$

for all $x, y \in \mathbb{R}$. In particular taking $x = 1$ we see that we must have $\nu \in \mathbb{Z}$, and if this is the case then (1.4) completely determines φ_ν, which is to say the multiplicity m_{π_ν} is one. Then by definition we have

$$R(f)\varphi_\nu = \left(\int_{\mathbb{R}} f(x) \exp(2i\pi\nu x)\, dx \right) \varphi_\nu = (\mathscr{F} f)(\nu)\varphi_\nu$$

and thus

$$\operatorname{tr}(\pi_\nu(f)) = (\mathscr{F} f)(\nu).$$

Hence in this case the trace formula reduces to the Poisson summation formula.

Theorem 1.3 (Poisson summation formula)**.** *Let f be a smooth, compactly supported function on $\mathbb{R}$. Then we have*

$$\sum_{\delta\in\mathbb{Z}} f(\delta) = \sum_{\nu\in\mathbb{Z}} (\mathscr{F} f)(\nu).$$

The fact that even in this simple setting the trace formula reduces to a non-trivial theorem should be seen as a hint about how deep and powerful it is in general. Selberg developped the trace formula for the group $G = \mathrm{GL}_2(\mathbb{R})$ (also for non-compact quotients). Since then many mathematicians have worked on traces formulas for more general groups, making them a cornerstone tool in the theory of automorphic forms and in the Langlands programme.

1.3. **The spherical transform.** Returning now to the general trace formula, calculating the traces $\mathrm{tr}(\pi(f))$ for arbitrary test functions f can challenging. In the example of $G = \mathbb{R}$ above, however, the space of $\pi(f)$ was one-dimensional – and $\pi(f)$ was given on this one-dimensional subspace by the Fourier transform. We will now see another situation where each operator $\pi(f)$ has rank (at most) one. Assume K is a compact subgroup of G such that for any irreducible unitary representation π of G the subspace π^K of π consisting of vectors fixed by the action of K is at most one-dimensional. This assumption might seem *ad hoc* but in view of the following theorem it is in fact very natural.

Theorem 1.4. *Let G be a semisimple connected real Lie group and let K be a maximal compact subgroup. Let π be an irreducible unitary representation of G. Then for each $\tau \in \widehat{K}$ the multiplicity of τ in the restriction $\pi_{|K}$ of π to K is at most* $\dim \tau$.

In particular taking τ to be the trivial representation of K, in this setting our assumption is satisfied. Now assume that f is *left*-K-invariant. Let $\pi \in \widehat{G}$ and let $V_\pi \subset L^2(X)$ be an invariant subspace isomorphic to π. Then for all $\varphi \in V_\pi$, for all $k \in K$ and almost all $x \in G$ we have

$$\begin{aligned}(\pi(f)\varphi)(xk) &= \int_G f(g)\varphi(xkg)\, d\mu(g) \\ &= \int_G f(k^{-1}g)\varphi(xg)\, d\mu(g) = (\pi(f)\varphi)(x)\end{aligned}$$

and so we see that $\pi(f)\varphi \in \pi^K$. Moreover it is easy to see that the adjoint of $\pi(f)$ is $\pi(f^*)$, where f^* is given by $f^*(g) = \overline{f(g^{-1})}$ for all $g \in G$. Thus if f is *right*-K-invariant then $\pi(f)$ vanishes on the orthogonal complement of π^K. Henceforth assume that f is right K-invariant. If $\pi^K = \{0\}$ then $\mathrm{tr}(\pi(f)) = 0$. Otherwise the space π^K is spanned by a single vector v_π, that we may assume to have norm 1. Then we have

$$\begin{aligned}\mathrm{tr}(\pi(f)) &= \langle \pi(f)v_\pi, v_\pi\rangle \\ &= \left\langle \int_G f(g)\pi(g)v_\pi \, d\mu(g), v_\pi \right\rangle = \int_G f(g)\phi_\pi(g)\, d\mu(g)\end{aligned}$$

where

$$\phi_\pi(g) = \langle \pi(g)v_\pi, v_\pi\rangle.$$

Since $v_\pi \in \pi^K$ and since $\langle \cdot, \cdot\rangle$ is G-invariant, it is to see that ϕ_π is itself left and right K-invariant. In this situation ϕ_π is called a *spherical function* for π, and the map $\hat{f} : \pi \mapsto \mathrm{tr}(\tilde{\pi}(f))$ is called the *spherical transform*. Here, $\tilde{\pi}$ is the contragredient representation to π, and the only reason we have put $\mathrm{tr}(\tilde{\pi}(f))$ instead of $\mathrm{tr}(\pi(f))$ is to have a formula that is formally analogous to the Fourier transform. An irreducible representation that has a non-trivial K-fixed vector is called *spherical*, whence the terminology.

The spectral decomposition of the space $L^2(X)$ (that is, the question of determining for which $\pi \in \widehat{G}$ we have $m_\pi \neq 0$) is quite a subtle matter, which involves deeply the arithmetic and geometric structure of Γ. The trace formula provides a valuable tool to study this question, provided that we can produce enough interesting test functions $\hat{f}$ on the spectral side, and that we can understand the geometric side – which would be the topic of a different course. It is therefore natural to ask the following questions.

Question 1. What kind of test functions $h = \hat{f}$ on $\widehat{G}$ can we produce?

Question 2. Given $h = \hat{f}$, can we recover f from h?

Before proceeding any further, let us note that using the left-K-invariance of ϕ_π we have $\mathrm{tr}(\pi(f)) = \mathrm{tr}(\pi(f^\#))$ where $f^\#(g) = \frac{1}{\mathrm{Vol}(K)} \int_K f(kg)\, d\mu_K(k)$. Since $f^\#$ is manifestly left-K-invariant we may assume without loss of generality that f is both left and right K-invariant.

1.4. **From representation theory to eigenfunctions.** Of course, to give a precise meaning to the above questions, one should first understand the structure of the unitary dual $\widehat{G}$ of G – which we will not discuss here. Now Questions 1 and 2 do not depend on the subgroup Γ any more, they are purely questions about the representation theory of G. In turn, these questions can be translated into questions of analysis. It should not be surprising that answering Questions 1 and 2 involves understanding the analytic behaviour of the spherical functions ϕ_π. These are examples of *matrix coefficients*. Given a unitary representation $\pi \in \widehat{G}$ and $u, v \in \pi$, define the matrix coefficient $\phi_{u,v}$ on G by

$$\phi_{u,v}(g) = \langle \pi(g)u, v\rangle.$$

In particular with earlier notations we have $\phi_\pi = \phi_{v_\pi, v_\pi}$. Observe that the map $T_v : u \mapsto \phi_{u,v}$ intertwines π with the regular right representation of G:

$$T_v(\pi(g)u)(x) = \langle \pi(xg)u, v\rangle = T_v(u)(xg) \qquad \text{for all } x, g \in G.$$

Thus we obtain a realisation of π in some space V_π of functions $G \to \mathbb{C}$. Now suppose we can find an operator D, acting on functions on G such that

(1) D preserves the G-invariant subspaces,
(2) D commutes with the regular right representation of K,

Then $D\phi_\pi \in V_\pi$ and $D\phi_\pi$ is right-K-invariant, thus since we are assuming $\dim(\pi^K) \leq 1$ we must have $D\phi_\pi = \lambda\phi_\pi$ for some $\lambda \in \mathbb{C}$. To construct an operator as above, given $g \in G$ define the operator R_g by $R_g f(x) = f(xg)$ for all $x \in G$. Then the operators of the form R_g obviously preserves G-invariant subspaces. The operator R_g does not in general commute with the action of K, but one can look for operators that do in the algebra generated by all the operators of the form R_g. In fact, in the setting of Lie groups, we have a differential structure on G, and we may consider *left-invariant differential operators*, which are in some sense limits of linear combinations of operators of the form R_g, and thus also preserve G-invariant subspaces. Instead of using the composition law of G, these have the advantage of involving the Lie algebra $\mathfrak{g}$ of G, which to some extent is easier to understand. Denoting by $D_K(G)$ the algebra of left-invariant differential operators that commute with the action of K, it follows from the above discussion that the matrix coefficients ϕ_π are joint eigenfunctions of every operator $D \in D_K(G)$ (there is a technicality here, namely that differential operators are only defined on *smooth* functions, but it turns out that ϕ_π is indeed smooth). As we will show in this course, in the setting of connected semisimple real Lie groups with finite centre, spherical functions are completely determined by their system of eigenvalues for the algebra $D_K(G)$, and moreover we shall be able to extract enough information about them to give the following answers to Question 1 and 2, which remarkably parallel Theorems 1.1 and 1.2. For the purpose of the introduction, we only give rather vague statements, and the more precise statements will be given at a further point in the course.

Theorem 1.5. *Let G be a connected semisimple real Lie group with finite centre, and let K be the subgroup consisting of the fixed points of the Cartan involution. There exists an explicit measure $\mu_{\widehat{G}}$ on $\widehat{G}$ satisfying the following property. Given a smooth,* *compactly function f on G that is left and right K-invariant, and given $\pi \in \widehat{G}$, define*

$$\hat{f}(\pi) = \int_G f(g)\phi_{\tilde{\pi}}(g)\, d\mu(g).$$

Then for all $g \in G$ we have

$$f(g) = \int_{\widehat{G}} \hat{f}(\pi)\phi_\pi(g)\, \mu_{\widehat{G}}(\pi).$$

Instead of considering only unitary representations of G, it is natural to consider the wider class of irreducible *admissible* representations of G. The natural notion of equivalence of admissible representations is called *infinitesimal equivalence*. We won't define these notions here, but we will mention that if π_1 and π_2 are two irreducible spherical admissible representations then we have $\phi_{\pi_1} = \phi_{\pi_2}$ if and only if π_1 and π_2 are infinitesimally equivalent. In general this is a looser condition than being isomorphic, however for unitary representations the two notions coincide. Let $A(G)$ be the set of infinitesimal equivalence classes of irreducible spherical admissible representations of G. The so-called Langlands classification shows (among other things) that there is an integer r and a finite group W_G called the *Weyl group* and a "natural" injective map $S : A(G) \to \mathbb{C}^r/W_G$ whose image is open and dense. Then we say a function h on $A(G)$ is Paley-Wiener if the function h_S defined on an open, dense subset of $\mathbb{C}^r$ by $h_S(z) = h(S^{-1}(W_G \cdot z))$ extends to a Paley-Wiener function on $\mathbb{C}^r$.

Theorem 1.6. *In the above setting, the spherical transform $f \mapsto \hat{f}$ is a bijection between the space of smooth, left and right K-invariant, compactly functions on G and the space of Paley-Wiener functions on $A(G)$.*

We hope these notes will illustrate how harmonic analysis of Lie groups fruitfully brings together geometry, algebra, analysis, and representation theory. Although this introduction was motivated by the later, we will not mention representation theory any more, working purely from the point of view of eigenfunctions of the algebra of differential operators. However representations are silently lurking in the background, and many of the results we shall cover have a natural interpretation in terms of representation theory. We invite the reader interested in this point of view to consult Knapp's book [Kna86].

The content of this course is primarily Harish-Chandra's work [HC58a, HC58b], and the results of Chevalley the latter is based on (some of which is unpublished and some appeared in [Che55]). A large part of these notes are taken from Helgason's books [Hel01], [Hel62], and particularly [Hel00]. Other excellent sources we have made use of are [Bum97], [Kna02], [Hum75], [Gar10] and [Wis01].

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2. Reminders from Differential Geometry

2.1. **Smooth manifolds and smooth maps.** Let $\mathscr{M}$ be a Hausdorff topological space. An **open chart** of dimension n on $\mathscr{M}$ is a pair (U, φ), where U is an open set of $\mathscr{M}$ and φ is an homeomorphism of U onto an open set of $\mathbb{R}^n$.

A C^∞**-manifold** (or smooth manifold) of dimension n is a Hausdorff topological space $\mathscr{M}$ together with a collection of open charts $(U_i, \varphi_i)_{i\in I}$ on $\mathscr{M}$ such that

- $\mathscr{M} = \bigcup_{i\in I} U_i$
- for each $i, j \in I$ the mapping $\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_j(U_i \cap U_j)$ is C^∞.

Remark 2.1. *In the definition of a manifold, further topological assumptions are often required. However we shall not need those.*

If $\mathscr{M}$ is a real C^∞-manifold a map $f : \mathscr{M} \to \mathbb{R}$ is **smooth** if for each local chart (U, φ) the map $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth. The set of smooth maps from $\mathscr{M}$ to $\mathbb{R}$ is denoted by $C^\infty(\mathscr{M})$. More generally, if $\mathscr{M}$ and $\mathscr{N}$ are two manifolds, a map $\phi : \mathscr{M} \to \mathscr{N}$ is smooth if $f \circ \phi \in C^\infty(\mathscr{M})$ for all $f \in C^\infty(\mathscr{N})$.

If $\mathscr{M}$ and $\mathscr{N}$ are two manifolds of dimensions m and n with open charts $(U_i, \varphi_i)_{i\in I}$, $(V_j, \psi_j)_{j\in J}$ respectively, the product manifold $\mathscr{M} \times \mathscr{N}$ is the manifold of dimension $m+n$ whose underlying topological space is $\mathscr{M} \times \mathscr{N}$ with the collection of open charts $(U_i \times V_j, \varphi_i \times \psi_j)_{\substack{i\in I\\ j\in J}}$, where $\varphi_i \times \psi_j$ is the map $U_i \times V_j \to \mathbb{R}^{m+n}$, $(x, y) \mapsto (\varphi_i(x), \psi(y))$.

2.2. **Vector fields and tangeant spaces.** If $\mathscr{M}$ is a real C^∞-manifold, a **vector field** on $\mathscr{M}$ is a linear map $X : C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M})$ satisfying Leibniz' law

$$X(fg) = fX(g) + X(f)g.$$

The set of all vector fields on $\mathscr{M}$ is denoted by $D^1(\mathscr{M})$. If $X, Y \in D^1(\mathscr{M})$ are two vector fields, their **Lie bracket** is

$$[X, Y] = X \circ Y - Y \circ X.$$

A tiny calculation shows that $[X, Y]$ satisfies the Leibniz law, and hence $[X, Y]$ is also a vector field. The Lie bracket satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in D^1(\mathscr{M})$. If $p \in \mathscr{M}$ and $X \in D^1(\mathscr{M})$ let X_p be the linear map $C^\infty(\mathscr{M}) \to \mathbb{R}$ defined by

$$X_p(f) = (Xf)(p).$$

The **tangent space** to $\mathscr{M}$ at p is the vector space

$$T_p\mathscr{M} = \{X_p : X \in D^1(\mathscr{M})\}.$$

Example 2.2. *Let V be a finite-dimensional vector space over $\mathbb{R}$. In particular V is a real manifold. For any $p \in V$ the tangent space T_pV can be identified with V itself by sending the vector $Y \in V$ to the element $\tilde{Y}_p$ of T_pV defined by*

$$\tilde{Y}_p(f) = (\tilde{Y}f)(p) = \left(\frac{d}{dt} f(p + tY)\right)_{t=0}.$$

for all $f \in C^\infty(V)$. The map $Y \mapsto \tilde{Y}$ is clearly injective. For the surjectivity assume without loss of generality that $V = \mathbb{R}^n$ and $p = 0$. Note that any $f \in C^\infty(V)$ can be written as

$$f(x) = \int_0^1 \frac{df}{dt}(tx)dt = \sum_{i=1}^n x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)dt}_{g_i(x)}.$$

Now given any $X \in D^1(\mathscr{M})$ we obtain

$$(Xf)(x) = \sum_{i=1}^n X(x_i g_i)(x) = \sum_{i=1}^n X(x_i)(x)g_i(x) + x_i X(g_i)(x),$$

and taking $x = 0$

$$X_0 f = \sum_{i=1}^{n} X_0(x_i)\frac{\partial f}{\partial x_i}(0) = \frac{df}{dt}(tY),$$

where $Y_i = X_0(x_i)$.

So far we have worked with vector fields followed by the evaluation at p, but an equivalent point of view is given by **derivations at** p, namely linear maps $\delta : C^\infty(\mathscr{M}) \to \mathbb{R}$ satisfying $\delta(fg) = f(p)\delta(g) + \delta(f)g(p)$.

Lemma 2.3. *Let* $\mathscr{M}$ *be a smooth manifold and* $p \in \mathscr{M}$*. Then* $T_p\mathscr{M}$ *is the space of derivations at* p*.*

Proof. It is clear from the definition of $T_p\mathscr{M}$ that its elements are derivations at p. So it remains to show that every derivation at p is of the form $\delta = X_p$ for some vector field X. Hence let δ be a derivation at p. Let (U, φ) be a local chart such that $p \in U$. let $h : \mathscr{M} \to \mathbb{R}$ be a smooth function such that $h(p) = 1$ and h vanishes outside of U (the existence of such a function is left as an exercise). Inspired by Example 2.2, let $Y = (\delta(x_i \circ \varphi))_i \in \mathbb{R}^m$, and let us define for all $f \in C^\infty(\mathscr{M})$

$$(Xf)(x) = h(x)\left(\frac{d}{dt} f \circ \varphi^{-1}(\varphi(x) + tY)\right)_{t=0}.$$

Note that a priori this definition only makes sense for $x \in U$, however the factor $h(x)$ ensures that the right hand side is zero for $x \notin U$. Firstly observe X satisfies the Leibniz law because $\frac{d}{dt}$ does. Thus $X \in D^1(\mathscr{M})$. Next we claim that $X_p = \delta$. Following the argument of Example 2.2, we have

$$\delta f = \sum_{i=1}^{n} \delta(x_i \circ \varphi)\frac{\partial f \circ \varphi^{-1}}{\partial x_i}(\varphi(p)) = (Xf)(p)$$

as desired. □

Let $\phi : \mathscr{M} \to \mathscr{N}$ be a smooth map between two smooth manifolds. Then ϕ induces a **pullback** map $\phi^* : C^\infty(\mathscr{N}) \to C^\infty(\mathscr{M}) : f \mapsto f \circ \phi$, and a map $d_p\phi : T_p\mathscr{M} \to T_{\phi(p)}\mathscr{N}, \delta \mapsto \delta \circ \phi^*$. Note that the fact $d_p\phi$ indeed takes values in $T_p\mathscr{N}$ follows from Lemma 2.3. When $\mathscr{M} = \mathbb{R}^m$ and $\mathscr{N} = \mathbb{R}^n$ this is the ordinary differential. Also observe that in the case $\mathscr{N} = \mathbb{R}$ then (upon identifying $T_{\phi(p)}\mathbb{R}$ with $\mathbb{R}$) $d_p\phi$ is given by the map $T_p\mathscr{M} \to \mathbb{R} : X \mapsto X(\phi)$.

Let $\mathscr{N}$ be a manifold. An **immersed submanifold** of $\mathscr{N}$ is a manifold $\mathscr{S}$ such that $\mathscr{S} \subset \mathscr{N}$ and the inclusion map $\iota_{\mathscr{S}}$ from $\mathscr{S}$ to $\mathscr{N}$ is smooth and its differential

$d_p\iota_{\mathscr{S}} : T_p\mathscr{S} \to T_p\mathscr{N}$ is injective for all $p \in \mathscr{S}$. In the sequel by **submanifold** we shall always mean an immersed submanifold.

Remark 2.4. *This definition is weaker than the definition of an embedded submanifold, which also requires the inclusion map to be a topological embedding.*

Later on we shall need the following technical lemmas.

Lemma 2.5. *Let $\mathscr{N}$ be a manifold and $\mathscr{S}$ be a submanifold of $\mathscr{N}$. Let $p \in \mathscr{S}$. Then there exists a local chart (V, κ) near p in $\mathscr{N}$ such that $\kappa(p) = 0$ and such that*

$$U = \{x \in V : \kappa_j(x) = 0 \text{ for } \dim \mathscr{S} < j \leq \dim \mathscr{N}\}$$

endowed with the restriction of $(\kappa_1, \cdots, \kappa_{\dim \mathscr{S}})$ to U is a local chart on $\mathscr{S}$ containing p.

Proof. Fix and let (U, φ) and (V, ψ) be local charts near p in $\mathscr{S}$ and $\mathscr{N}$ respectively, such that $\varphi(p) = \psi(p) = 0$. Let $F = (F_1, \cdots, F_{\dim \mathscr{N}})$ be defined in a neighbourhood of 0 in $\mathbb{R}^{\dim \mathscr{S}}$ by $F = \psi \circ \varphi^{-1}$. Since $d_p\iota_{\mathscr{S}}$ is injective the Jacobian matrix $\left(\frac{\partial F_j}{\partial x_i}(0)\right)_{\substack{1 \leq i \leq \dim \mathscr{S} \\ 1 \leq j \leq \dim \mathscr{N}}}$ has rank $\dim \mathscr{S}$. Hence without loss of generality we may assume that the matrix $J = \left(\frac{\partial F_j}{\partial x_i}(0)\right)_{1 \leq i,j \leq \dim \mathscr{S}}$ is invertible. Thus in a neighbourhood of 0 the map $(F_1, \cdots, F_{\dim \mathscr{S}})$ has inverse map G. That is, in a neighbourhood of p in $\mathscr{S}$ we have $G(\psi_1, \cdots, \psi_{\dim \mathscr{S}}) = \varphi$. Now put

$$\kappa_j = \begin{cases} \psi_j & \text{if } 1 \leq j \leq \dim \mathscr{S}, \\ \psi_j - F_j(G(\psi_1, \cdots, \psi_{\dim \mathscr{S}})) & \text{if } j > \dim \mathscr{S}. \end{cases}$$

It remains to see that $(\kappa_1, \cdots, \kappa_{\dim \mathscr{S}}) \circ \varphi^{-1}$ and $\kappa \circ \psi^{-1}$ are local diffeomorphisms. The Jacobian matrix of $(\kappa_1, \cdots, \kappa_{\dim \mathscr{S}}) \circ \varphi^{-1}$ is $\left(\frac{\partial \psi_j \circ \varphi^{-1}}{\partial x_i}(0)\right)_{1 \leq i,j \leq \dim \mathscr{S}} = J$ which is invertible. On the other hand the Jacobian matrix of $\kappa \circ \psi^{-1}$ is of the form $\begin{bmatrix} \mathrm{Id}_{\mathbb{R}^{\dim \mathscr{S}}} & * \\ & \mathrm{Id}_{\mathbb{R}^{\dim \mathscr{N} - \dim \mathscr{S}}} \end{bmatrix}$ and hence also invertible. □

Lemma 2.6. *Let $\mathscr{N}$ be a manifold and $\mathscr{S}$ be a submanifold of $\mathscr{N}$. Let $\mathscr{M}$ be a manifold and let $\phi : \mathscr{M} \to \mathscr{N}$ be a smooth map such that $\varphi(\mathscr{M}) \subset \mathscr{S}$. If the map $\phi : \mathscr{M} \to \mathscr{S}$ is continuous, then it is smooth.*

Proof. Let $p \in \mathscr{M}$. consider the local chart (V, κ) near $\phi(p)$ and the open set $U \subset \mathscr{S}$ given by Lemma 2.5 Since ϕ is continuous $\phi^{-1}(U)$ is an open set containing p and hence we may find a local chart (W, φ) near p such that $W \subset \phi^{-1}(U)$. In particular $\phi(W) \subset U$. Since $\phi : \mathscr{M} \to \mathscr{N}$ is smooth, the map $\kappa \circ \phi : W \to \mathbb{R}^{\dim \mathscr{N}}$ is also smooth. But then so is the map $(\kappa_1, \cdots, \kappa_{\dim \mathscr{S}}) \circ \phi : W \to \mathbb{R}^{\dim \mathscr{S}}$, from which it follows that $\phi : \mathscr{M} \to \mathscr{S}$ is smooth. □

Remark 2.7. *The hypothesis that* $\phi : \mathscr{M} \to \mathscr{S}$ *be continuous is not automatic, because there might be open sets of* $\mathscr{S}$ *that are not the intersection of* $\mathscr{S}$ *with an open set of* $\mathscr{N}$*. In fact, let us construct an example of a smooth map* $\phi : \mathscr{M} \to \mathscr{N}$ *such that* $\phi : \mathscr{M} \to \mathscr{S}$ *is not continuous. Fix* $f : \mathbb{R} \to \mathbb{R}$ *such that* f *is smooth,* $f(t) > 0$ *for* $t > 0$ *and* $f(t) = 0$ *for* $t \leq 0$*. Consider* $R_1 = \{(t, f(t)) : t > 0\} \subset \mathbb{R}^2$ *and* $R_2 = \{(t, 0) : t \in \mathbb{R}\} \subset \mathbb{R}^2$*, both endowed with the subspace topology. Let* $\mathscr{S} = R_1 \cup R_2$ *endowed with the disjoint union topology.* $\mathscr{S}$ *is indeed a submanifold of* $\mathbb{R}^2$*. Finally let* $\phi : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t, f(t))$*. Then* $\phi : \mathbb{R} \to \mathbb{R}^2$ *is smooth, and has its image contained in* $\mathscr{S}$*. On the other hand* R_2 *is an open subset of* $\mathscr{S}$*, and* $\phi^{-1}(R_2) = [0, \infty)$ *is not open, hence* $\phi : \mathbb{R} \to \mathscr{S}$ *is not continuous.*

3. Lie groups and Lie algebras

A (real) **Lie group** is a group G which is also a C^∞ manifold such that the maps

$$G \times G \to G, (x, y) \mapsto xy \quad \text{and} \quad G \to G, x \mapsto x^{-1}$$

are smooth. In the sequel we shall denote the neutral element of a group G by e_G, or just e if G is clear from the context. A **morphism** of Lie groups is a morphism of groups which is also a smooth map.

Example 3.1. *The group* $\mathrm{GL}_2(\mathbb{R})$ *of* 2×2 *matrices non-zero determinant is an open subset of* $\mathbb{R}^4$*, which immediately endows it with a structure of* C^∞ *manifold. The multiplication and inverse are smooth because they are given by rational functions.*

Example 3.2. *The group* $\mathrm{SL}_2(\mathbb{R})$ *of* 2×2 *matrices with determinant* 1 *can be viewed as a subset of* $\mathbb{R}^4$*, and thereby endowed with the subset topology. We can turn it into a smooth manifold by specifying the local charts*

$$U_1 = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R}) : a \neq 0\}, \quad \varphi_1 : U_1 \to \mathbb{R}^3, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto (a, b, c)\,,$$
$$U_2 = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R}) : b \neq 0\}, \quad \varphi_2 : U_2 \to \mathbb{R}^3, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto (a, b, d)\,.$$

The multiplication and inverse are smooth for the same reason as in the case of $\mathrm{GL}_2(\mathbb{R})$*.*

Let G be a Lie group and for $g \in G$ let ℓ_g and ρ_g denote the then the left and right translation of G on itself respectively, that is

$$\ell_g : G \to G, x \mapsto gx \quad \text{and} \quad \rho_g : G \to G, x \mapsto x \mapsto xg.$$

They in turn induce corresponding actions on smooth functions $f : G \to \mathscr{M}$ is smooth, where $\mathscr{M}$ is any smooth manifold. Namely for all $g \in G$ the functions

$$L_g f : x \mapsto f(gx) \quad \text{and} \quad R_g f : x \mapsto f(xg)$$

are themselves smooth. Observe that

$$L_g \circ L_x = L_{xg} \quad \text{and} \quad R_g \circ R_x = R_{gx}.$$

In the notations of § 2.2 we have $L_g = \ell_g^*$ and $R_g = \rho_g^*$, and in particular the map $d_e\ell_g : T_eG \to T_gG$ is given by $\mathsf{x} \mapsto \mathsf{x} \circ L_g$ (and similarly for R_g).

A vector field X on G is called **left-invariant** if $L_g \circ X = X \circ L_g$ for all $g \in G$. Let $\mathscr{L}(G)$ be the vector space of left-invariant vector fields on G. If $X, Y \in \mathscr{L}(G)$ then $[X, Y] = X \circ Y - Y \circ X \in \mathscr{L}(G)$ (we already know that $[X, Y]$ is a vector field, and the left-invariance is obvious). The vector space $\mathscr{L}(G)$ endowed with the Lie bracket is called the **Lie algebra of** G. It is indeed a Lie algebra according to the following definition.

Definition 3.3. A (real) Lie algebra is a real vector $\mathfrak{a}$ space endowed with a bilinear map $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ satisfying the following

(1) $[X, X] = 0$ for all $X \in \mathfrak{a}$,
(2) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{a}$.

Remark 3.4. *Axiom (2) is called the Jacobi identity. Using bilinearity and Axiom (1) it is easy to see that $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{a}$.*

The Lie algebra $\mathfrak{a}$ is said to be **commutative** is the Lie bracket $[\cdot,\cdot]$ is identically zero. A **Lie subalgebra** is a vector subspace $\mathfrak{b}$ of $\mathfrak{a}$ such that $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}$. A **morphism of Lie algebras** is a linear map $\sigma : \mathfrak{a} \to \mathfrak{a}'$ between Lie algebras such that $[\sigma(X), \sigma(Y)] = \sigma([X, Y])$ for all $X, Y \in \mathfrak{a}$.

Remark 3.5. *In general $X \circ Y$ is* **not** *a vector field (it does not satisfy the Leibniz law). That is, $\mathscr{L}(G)$ is a Lie algebra for the Lie bracket given by $[X, Y] = X \circ Y - Y \circ X$, but its is not an algebra for $\circ$.*

Remark 3.6. *If G is a commutative Lie group then $\mathfrak{g}$ is a commutative Lie algebra, justifying the terminology. This can already be proved using local charts, or this follows directly from (3.9) below. More generally, we will see in Proposition 3.7 below that left-invariant vector fields X may be identified with tangent vectors x to G. Geometrically, one can think of a tangent vector x at $e_G \in G$ as an infinitesimal variation around e_G in some direction X. Call g_{x} the corresponding element of G. Then if y is another tangent vector, the elements $g_{\mathsf{x}}g_{\mathsf{y}}$ and $g_{\mathsf{y}}g_{\mathsf{x}}$ are both given in first approximation by $g_{\mathsf{x}+\mathsf{y}}$ and thus ${g_{\mathsf{x}}g_{\mathsf{y}}g_{\mathsf{x}}}^{-1}{g_{\mathsf{y}}}^{-1}$ equals to first order e_G. Then the bracket $[X, Y]$ gives the second order term in this approximation. See Example 3.10 below for a more precise discussion.*

We now compare $\mathscr{L}(G)$, as a vector space, to the tangent space T_eG of G at e, that we usually denote by $\mathfrak{g}$. The following result should not be surprising: a vector field on G is the smooth choice of a tangent vector at each point x of G. Now for a left-invariant vector field, this choice of tangent vector at gx is determined (via left translation) by the choice of tangent vector at x. Since G acts transitively on itself, a left-invariant vector field is determined by the choice of a tangent vector at a single point.

Proposition 3.7. *The map $\theta : \mathscr{L}(G) \to \mathfrak{g}, X \mapsto X_e$ is a vector space isomorphism.*

Proof. We construct an inverse map $\eta : \mathfrak{g} \to \mathscr{L}(G)$ as follows. If $\mathsf{x} \in \mathfrak{g} = T_eG$ then $\eta(\mathsf{x})$ is the "right convolution by x", namely the vector field

$$\begin{aligned} C^\infty(G) &\to C^\infty(G) \\ f &\mapsto f * \mathsf{x}, \end{aligned}$$

where

$$(f * \mathsf{x})(g) = \mathsf{x}(L_g f).$$

for all $f \in C^\infty(G)$. First, we need to check $\eta(\mathsf{x})$ is indeed a left-invariant vector field for all $\mathsf{x} \in T_eG$. The Leibniz law is easily checked. For the left-invariance, we need to check $L_y \circ \eta(\mathsf{x}) = \eta(\mathsf{x}) \circ L_y$ for all $y \in G$, that is

$$L_y(f * \mathsf{x}) = (L_y f) * \mathsf{x}$$

for all $f \in C^\infty(G)$. We have for all $g \in G$

$$\begin{aligned} (L_y(f * \mathsf{x}))(g) &= (f * \mathsf{x})(yg) \\ &= \mathsf{x}(L_{yg} f) \\ &= \mathsf{x}(L_g \circ L_y f) \\ &= \mathsf{x}(L_y f)(g), \end{aligned}$$

as desired. Next we prove that $\eta \circ \theta = \mathrm{Id}_{\mathscr{L}(G)}$ and $\theta \circ \eta = \mathrm{Id}_{\mathfrak{g}}$. Indeed, let $X \in \mathscr{L}(G)$ then we have for all $f \in C^\infty(G)$ and for all $g \in G$

$$\begin{aligned} (\eta \circ \theta(X)(f))(g) &= (f * \theta(X))(g) = X_e(L_g f) \\ &= X(L_g f)(e) \\ &= L_g(Xf)(e) \\ &= (Xf)(g), \end{aligned}$$

as desired. On the other hand if $\mathsf{x} \in T_e(G)$ we have

$$(\theta \circ \eta(\mathsf{x}))(f) = (f * \mathsf{x})(e) = \mathsf{x}(L_e f) = \mathsf{x}(f),$$

and the proof is complete. □

As a result, we can identify the Lie algebra of G to the vector space $\mathfrak{g}$, endowed with the Lie bracket defined by $[\theta(X), \theta(Y)] = \theta([X, Y])$.

Proposition 3.8. *Let $\phi : G \to G'$ be a morphism of Lie groups. Then $d_{e_G}\phi : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras.*

Remark 3.9. *This says that the assignment $G \mapsto \mathfrak{g}$ is a functor from the category of Lie groups to the category of Lie algebras.*

Proof. We first claim that for all $f \in C^\infty(G')$ and for all $\mathsf{x} \in \mathfrak{g}$ we have

$$(f \circ \phi) * \mathsf{x} = (f * d_{e_G}\phi(\mathsf{x})) \circ \phi.$$

Indeed for all $g \in G$ we have by definition

$$((f \circ \phi) * \mathsf{x})(g) = \mathsf{x}(L_g(f \circ \phi))$$

and

$$\begin{aligned}((f * d_{e_G}\phi(\mathsf{x})) \circ \phi)(g) &= (d_{e_G}\phi(\mathsf{x}))(L_{\phi(g)}f) \\ &= \mathsf{x}((L_{\phi(g)}f) \circ \phi),\end{aligned}$$

and the claim follows upon observing that $L_g(f \circ \phi) = (L_{\phi(g)}f) \circ \phi$. Now for $\mathsf{x}, \mathsf{y} \in \mathfrak{g}$ let $\mathsf{x}' = d_{e_G}\phi(\mathsf{x})$ and $\mathsf{y}' = d_{e_G}\phi(\mathsf{y})$. Then we have

$$\begin{aligned}[\mathsf{x}', \mathsf{y}'](f) &= (f * \mathsf{y}' * \mathsf{x}')(e_{G'}) - (f * \mathsf{x}' * \mathsf{y}')(e_{G'}) \\ &= \mathsf{x}'(f * \mathsf{y}') - \mathsf{y}'(f * \mathsf{x}') \\ &= \mathsf{x}((f * \mathsf{y}') \circ \phi) - \mathsf{y}((f * \mathsf{x}') \circ \phi) \\ &= \mathsf{x}((f \circ \phi) * \mathsf{y}) - \mathsf{y}((f \circ \phi) * \mathsf{x}),\end{aligned}$$

where we have used the claim in the last step. On the other hand

$$\begin{aligned}(d_{e_G}[\mathsf{x}, \mathsf{y}])(f) &= (f \circ \phi) * [\mathsf{x}, \mathsf{y}] \\ &= (f * \mathsf{y} * \mathsf{x})(e_G) - (f * \mathsf{x} * \mathsf{y})(e_G) \\ &= \mathsf{x}((f \circ \phi) * \mathsf{y}) - \mathsf{y}((f \circ \phi) * \mathsf{x}),\end{aligned}$$

and so we have shown

$$[d_{e_G}\phi(\mathsf{x}), d_{e_G}\phi(\mathsf{y})](f) = d_{e_G}([\mathsf{x}, \mathsf{y}])(f)$$

for all $f \in C^\infty(G')$. □

Example 3.10. *Since $\mathrm{GL}_2(\mathbb{R})$ is an open subset of $\mathrm{Mat}_2(\mathbb{R})$, its Lie algebra is $\mathfrak{gl}_2 = \mathrm{Mat}_2(\mathbb{R})$. More precisely, if $\mathsf{x} \in \mathrm{Mat}_2(\mathbb{R})$ then (recall Example 2.2) we can view x as the element of $\mathfrak{gl}_2$ defined by*

$$\mathsf{x}f = \left(\frac{d}{dt}f(I_2 + t\mathsf{x})\right)_{t=0}$$

for all $f \in C^\infty(\mathrm{GL}_2(\mathbb{R}))$. To calculate the Lie bracket, we can use the map η from the proof of Proposition 3.7. Namely, the image of x by η is the vector field defined by

$$(f * \mathsf{x})(g) = \mathsf{x}(L_g f) = \left(\frac{d}{dt}f(g + tg\mathsf{x})\right)_{t=0}.$$

So we need to calculate

$$(\eta(\mathsf{x}) \circ \eta(\mathsf{y}) - \eta(\mathsf{y}) \circ \eta(\mathsf{x}))\,(f) = f * \mathsf{y} * \mathsf{x} - f * \mathsf{x} * \mathsf{y}.$$

It is useful to notice that the Taylor expansion of f around g is given by

$$f(g(1+t\mathsf{x})) = f(g) + t(f * \mathsf{x})(g) + \frac{t^2}{2}\left(\frac{d^2}{dt^2} f(g+tg\mathsf{x})\right)_{t=0} + O(t^3)$$

as $t \to 0$. Thus we have

$$\begin{aligned} f(g(1+t\mathsf{y})(1+t\mathsf{x})) = {} & f(g) + t(f * \mathsf{x})(g) + t(f * \mathsf{y})(g) \\ & + t^2(f * \mathsf{y} * \mathsf{x})(g) \\ & + \frac{t^2}{2}\left(\frac{d^2}{dt^2} f(g+tg\mathsf{x})\right)_{t=0} + \frac{t^2}{2}\left(\frac{d^2}{dt^2} f(g+tg\mathsf{y})\right)_{t=0} + O(t^3). \end{aligned}$$

Similarly

$$\begin{aligned} f(g(1+t\mathsf{x})(1+t\mathsf{y})) = {} & f(g) + t(f * \mathsf{x})(g) + t(f * \mathsf{y})(g) \\ & + t^2(f * \mathsf{x} * \mathsf{y})(g) \\ & + \frac{t^2}{2}\left(\frac{d^2}{dt^2} f(g+tg\mathsf{x})\right)_{t=0} + \frac{t^2}{2}\left(\frac{d^2}{dt^2} f(g+tg\mathsf{y})\right)_{t=0} + O(t^3). \end{aligned}$$

Subtracting, we obtain

$$\begin{aligned} t^2(f * \mathsf{y} * \mathsf{x} - f * \mathsf{x} * \mathsf{y})(g) = {} & f(g(1+t\mathsf{y})(1+t\mathsf{x})) \\ & - f(g(1+t\mathsf{x})(1+t\mathsf{y})) + O(t^3). \end{aligned} \tag{3.1}$$

But

$$(1+t\mathsf{x})(1+t\mathsf{y}) = (1+t\mathsf{y})(1+t\mathsf{x}) + t^2(\mathsf{xy} - \mathsf{yx})$$

and thus

$$\begin{aligned} f(g(1+t\mathsf{y})(1+t\mathsf{x})) &= f(g(1+t\mathsf{x})(1+t\mathsf{y})) + t^2(f * (\mathsf{xy} - \mathsf{yx}))(g(1+t\mathsf{y})(1+t\mathsf{x})) + O(t^4) \\ &= f(g(1+t\mathsf{x})(1+t\mathsf{y})) + t^2(f * (\mathsf{xy} - \mathsf{yx}))(g) + O(t^3) \end{aligned}$$

Reporting this in (3.1) we obtain

$$t^2(f * \mathsf{y} * \mathsf{x} - f * \mathsf{x} * \mathsf{y})(g) = t^2(f * (\mathsf{xy} - \mathsf{yx}))(g) + O(t^3),$$

from which we deduce $f * \mathsf{y} * \mathsf{x} - f * \mathsf{x} * \mathsf{y} = f * (\mathsf{xy} - \mathsf{yx})$ and hence for $\mathsf{x}, \mathsf{y} \in \mathfrak{gl}_2$ the Lie bracket is simply given by $[\mathsf{x}, \mathsf{y}] = \mathsf{xy} - \mathsf{yx}$.

Example 3.11. *The subgroup $\mathrm{SL}_2(\mathbb{R})$of $\mathrm{GL}_2(\mathbb{R})$ is defined by the equation $\det g = 1$. Taking the differential of this equation at $g = e$, we deduce that $\mathfrak{sl}_2$ is the space of 2×2 matrices whose trace equals zero. Let $\iota : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R})$ be the inclusion. Then the map $d_e\iota$ is just the inclusion of $\mathfrak{sl}_2$ in $\mathfrak{gl}_2$, and in particular the Lie bracket on $\mathfrak{sl}_2$ is still given by $[\mathsf{x}, \mathsf{y}] = \mathsf{xy} - \mathsf{yx}$. Needless to say, the same argument works more generally for Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$.*

3.1. The exponential map.

Proposition 3.12. *Let G be a Lie group and let $\mathsf{x} \in \mathfrak{g}$. Then there exists a unique morphism of Lie groups $\theta_{\mathsf{x}} : \mathbb{R} \to G$ such that $\theta_{\mathsf{x}}'(0) = \mathsf{x}$.*

Proof sketch. We start with the uniqueness. Since θ_{x} is a morphism of groups we have

$$\theta_{\mathsf{x}}(s+t) = \theta_{\mathsf{x}}(s)\theta_{\mathsf{x}}(t) \tag{3.2}$$

for all $s, t \in \mathbb{R}$. Differentiating (3.2) at $t = 0$ and applying the chain rule we obtain $\theta_{\mathsf{x}}'(s) = \mathsf{x} \circ L_{\theta(s)}$ for all $s \in \mathbb{R}$. Hence θ_{x} is a solution of the system of ordinary differential equations (satisfying a certain Lipschitz condition) with initial conditions

$$\begin{cases} \gamma'(s) = \mathsf{x} \circ L_{\gamma(s)} \\ \gamma(0) = e \end{cases} \tag{3.3}$$

This proves uniqueness, as well as existence of a solution θ_{x} to (3.3) defined in a neighbourhood of 0. We use (3.2) to extend this solution to all of $\mathbb{R}$. Thus we have proved that for all $\mathsf{x} \in \mathfrak{g}$ there exists a unique $\theta_{\mathsf{x}} : \mathbb{R} \to G$ satisfying (3.3). It remains to show θ_{x} satisfies (3.2). But for each fixed $s \in \mathbb{R}$ the curve given by $\gamma : t \mapsto \theta(s)^{-1}\theta(s+t)$ also satisfies (3.3) and hence $\gamma = \theta_{\mathsf{x}}$, which was the desired result. □

Remark 3.13. *Geometrically, $\theta_{\mathsf{x}}(t)$ is given by following the geodesic passing through e_G with tangent vector x up to time t.*

Let G be a Lie group. We define the **exponential** of $\mathsf{x} \in \mathfrak{g}$ as $\exp(\mathsf{x}) = \theta_{\mathsf{x}}(1)$, where θ_{x} is given by Proposition 3.12. It satisfies

$$\exp((s+t)\mathsf{x}) = \exp(s\mathsf{x})\exp(t\mathsf{x}), \tag{3.4}$$

which follows from the fact that for all $\lambda \in \mathbb{R}$ we have $\theta_{\mathsf{x}}(\lambda t) = \theta_{\lambda\mathsf{x}}(t)$, as can be checked by replacing x with $\lambda\mathsf{x}$ in (3.3).

Remark 3.14. *If $G = \mathrm{GL}_n(\mathbb{R})$ (or some Lie subgroup thereof), so that $\mathfrak{g} = \mathrm{Mat}_n(\mathbb{R})$ (or a subalgebra) then the exponential map is just the usual matrix exponential given by*

$$\exp(\mathsf{x}) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathsf{x}^k.$$

In Proposition 3.8 we have already seen that if $\phi : G \to G'$ is a morphism of Lie groups then $d_{e_G}\phi : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras. We supplement this result with the following.

Lemma 3.15. *Let $\phi : G \to G'$ be a morphism of Lie groups. Then for all $\mathsf{x} \in \mathfrak{g}$ we have*

$$\phi(\exp(\mathsf{x})) = \exp(d_{e_G}\phi(\mathsf{x})).$$

Proof. Let $\mathsf{x} \in \mathfrak{g}$. Then the map $\theta : \mathbb{R} \to G' : t \mapsto \phi(\exp(t\mathsf{x}))$ is a morphism of Lie groups satisfying $\theta'(0) = d_{e_G}\phi(\mathsf{x})$. Thus by Proposition 3.12 and by definition of the exponential we have $\exp(d_{e_G}\phi(\mathsf{x})) = \theta(1) = \phi(\exp(\mathsf{x}))$ as claimed. □

Proposition 3.16. *Let G be a Lie group. There exists an open neighbourhood N_0 of 0 in $\mathfrak{g}$ and an open neighbourhood N_e of e such that the exponential map induces a diffeomorphism from N_0 to N_e.*

Proof. Let (U, φ) be a local chart around e, such that $\varphi(e) = 0$. Consider the map $\psi : (t, \mathsf{x}) \mapsto \varphi \circ \theta_{\mathsf{x}}(t)$, where θ_{x} is the map from Proposition 3.12. Fix $t > 0$. Then for all sufficiently small s we have

$$\psi(st, \mathsf{x}) = \psi(t, s\mathsf{x}). \tag{3.5}$$

Denote by D_0 the derivative with respect to the first coordinate, and observe that by definition we have

$$(D_0\psi)(0, \mathsf{x}) = d_e\varphi(\mathsf{x}). \tag{3.6}$$

Fix a basis $(b_1, \cdots, b_n)$ of $\mathfrak{g}$ and for $1 \leq i \leq n = \dim(G)$ denote by x_i the i-th coordinate, and by D_i the derivative with respect to x_i. Differentiating (3.5) twice with respect to s we obtain

$$t^2(D_0^2\psi)(st, \mathsf{x}) = \sum_{i,j=1}^{n} \mathsf{x}_i\mathsf{x}_j(D_iD_j\psi)(t, s\mathsf{x}). \tag{3.7}$$

Taylor expansion and the mean value theorem imply that there exists $t^* \in [0, t]$ such that

$$\psi(t, \mathsf{x}) = t(D_0\psi)(0, \mathsf{x}) + \frac{t^2}{2}(D_0^2)(t^*, \mathsf{x}). \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we obtain

$$\psi(t, \mathsf{x}) = td_e\varphi(\mathsf{x}) + \frac{1}{2}\sum_{i,j=1}^{n} \mathsf{x}_i\mathsf{x}_j(D_iD_j\psi)\left(t, \frac{t^*}{t}\mathsf{x}\right).$$

This shows that the map $\mathsf{x} \mapsto \psi(t, \mathsf{x})$ is differentiable, and its differential is given by $td_e\varphi$, which is invertible for $t \neq 0$. In particular taking $t = 1$ establishes the result. □

Corollary 3.17. *Let G be a connected Abelian Lie group. Then* exp *is an surjective morphism of Lie groups between the group* $(\mathfrak{g},+)$ *and* G.

Proof. Since G is commutative, it is easy to verify that given $\mathsf{x},\mathsf{y}\in\mathfrak{g}$ the map $\theta:\mathbb{R}\to G: t\mapsto\exp(t\mathsf{x})\exp(t\mathsf{y})$ is a morphism of Lie groups. One verifies that $\theta'(0)=\mathsf{x}+\mathsf{y}$, and thus by definition of the exponential we have $\exp(\mathsf{x})\exp(\mathsf{y})=\theta(1)=\exp(\mathsf{x}+\mathsf{y})$ as claimed. The surjectivity follows from Proposition 3.16 and the fact that if G is a connected topological group and if H is a subgroup containing a non-empty open subset of G then $H=G$. □

3.2. **Differential operators.** Proposition 3.7 allows us to identify the Lie algebra $\mathfrak{g}$ of a Lie group G with left-invariant vector fields. These are given in local charts by some differential operators, but we can now make things more precise and coordinate-free. Namely, let $X\in\mathscr{L}(G)$. By definition we have $\left(\frac{d}{dt}\exp(tX_e)\right)_{t=0}=X_e$ and thus for any $f\in C^\infty(G)$

$$\begin{aligned}\left(\frac{d}{dt}f(g\exp tX_e)\right)_{t=0} &= \left(\frac{d}{dt}(L_gf)(\exp tX)\right)_{t=0}\\ &= d_e(L_gf)(X_e)\\ &= X_e(L_gf)\\ &= L_g(X_ef),\end{aligned}$$

and thus we have obtained the formula

$$(Xf)(g)=\left(\frac{d}{dt}f(g\exp tX_e)\right)_{t=0}. \tag{3.9}$$

We let $D(G)$ be the algebra of **left-invariant differential operators** on $C^\infty(G)$, that is the algebra over $\mathbb{R}$ generated by all the left-invariant vector fields on G as well as the identity. A classical result is that $D(G)$ is isomorphic to the **universal enveloping algebra** of $\mathfrak{g}$, but we won't prove this here (or indeed define the universal enveloping algebra).

We will however see that there is a linear isomorphism between $D(G)$ (as a vector space) and the underlying vector space of the so-called **symmetric algebra** $S(\mathfrak{g})$. If V is a finite-dimensional real vector space, a polynomial function on V is a real-valued function on V given by $v\mapsto P(\ell_1(v),\cdots,\ell_m(v))$ where $\ell_1,\cdots,\ell_m$ are linear forms on V, and $P\in\mathbb{R}[X_1,\cdots,X_m]$ for some integer m. Then the symmetric algebra $S(V)$ is the commutative algebra of polynomial functions on the dual V^* of V. If $X_1,\cdots,X_n$, is a basis of V, then identifying V with its bidual, the symmetric algebra may be identified with $\mathbb{R}[X_1,\cdots,X_n]$. Fix a basis $X_1,\cdots,X_n$ of $\mathscr{L}(G)$. If

$\bar{t} = (t_1, \cdots, t_n) \in \mathbb{R}^n$, let $X(\bar{t}) = t_1 X_1 + \cdots + t_n X_n$ and if $\overline{m} = (m_1, \cdots, m_n)$ is a tuple of non-negative integers let $m = m_1 + \cdots + m_n$ and let $X_{\overline{m}}$ be the coefficient of $t_1^{m_1} \cdots t_n^{m_n}$ in $X(\bar{t})^m$.

Lemma 3.18. *The elements $X_{\overline{m}}$, where $\overline{m} = (m_1, \cdots, m_n)$ is a tuple of non-negative integers are linearly independent.*

Proof. Using (3.4), (3.9) and induction, we have for all $X \in \mathscr{L}(G)$, for all $f \in C^\infty(G)$ and for all integer $m \geq 0$

$$(X^m f)(g) = \left(\frac{d^m}{dt^m} f(g \exp tX_e) \right)_{t=0}. \tag{3.10}$$

On the other hand if $h : \mathbb{R}^n \to \mathbb{C}$ is any smooth function and $F(t) = h(tt_1, \cdots, tt_n)$ then by induction we have

$$\frac{d^m}{dt^m} F(t) = \sum_{m_1 + \cdots + m_n = m} \frac{m!}{m_1! \cdots m_n!} t_1^{m_1} \cdots t_n^{m_n} (D_1^{m_1} \cdots D_n^{m_n} h)(tt_1, \cdots tt_n), \tag{3.11}$$

where D_i is the derivative with respect to the i-th variable. Now applying the above with $X = X(\bar{t}) = t_1 X_1 + \cdots + t_n X_n$ and $h(t) = f(g \exp tX_e)$, we obtain

$$(X(\bar{t})^m f)(g) = \sum_{m_1 + \cdots + m_n = m} t_1^{m_1} \cdots t_n^{m_n} X_{\overline{m}} f(g) =$$
$$\sum_{m_1 + \cdots + m_n = m} \frac{m!}{m_1! \cdots m_n!} t_1^{m_1} \cdots t_n^{m_n} \left(\frac{\partial^m}{\partial t_1^{m_1} \cdots \partial t_n^{m_n}} f(g \exp X(\bar{t})_e) \right)_{t_1 = \cdots = t_n = 0}.$$

Identifying the coefficients of $t_1^{m_1} \cdots t_n^{m_n}$ we deduce

$$(X_{\overline{m}} f)(g) = \frac{m!}{m_1! \cdots m_n!} \left(\frac{\partial^m}{\partial t_1^{m_1} \cdots \partial t_n^{m_n}} f(g \exp X(\bar{t})_e) \right)_{t_1 = \cdots = t_n = 0}. \tag{3.12}$$

By Proposition 3.16, this proves that the elements $X_{\overline{m}}$ are linearly independent. □

Corollary 3.19. *The following are basis of $D(G)$*

(1) the elements $X_1^{e_1} \cdots X_n^{e_n}$, where $e_1, \cdots, e_n$ are non-negative integers,
(2) the elements $X_{\overline{m}}$, where $\overline{m} = (m_1, \cdots, m_n)$ is a tuple of non-negative integers.

Proof. By definition, $D(G)$ is spanned by expressions of the form

$$X_1^{e_{11}} \cdots X_n^{e_{n1}} \cdots X_1^{e_{1r}} \cdots X_n^{e_{nr}}.$$

Using the fact that $X_iX_j = [X_i, X_j] + X_jX_i$ and $[X_i, X_j] \in \mathscr{L}(G)$, each such expression can be written as a linear combination of elements $X_1^{e_1} \cdots X_n^{e_n}$ with

$$\sum_{i=1}^{n} e_i \leq \sum_{i=1}^{n} \sum_{j=1}^{r} e_{ij}. \tag{3.13}$$

In particular (3.13) implies that $X_{\overline{m}} \in \mathrm{Span}\{X_1^{e_1} \cdots X_n^{e_n} : e_1 + \cdots + e_n \leq m\}$ and thus

$$\mathrm{Span}\{X_{\overline{m}} : m_1 + \cdots + m_n \leq m\} \subset \mathrm{Span}\{X_1^{e_1} \cdots X_n^{e_n} : e_1 + \cdots + e_n \leq m\}. \tag{3.14}$$

Since

$$\#\{X_1^{e_1} \cdots X_n^{e_n} : e_1 + \cdots + e_n \leq m\} \leq \#\{X_{\overline{m}} : m_1 + \cdots + m_n \leq m\}$$

and the set on the right hand side consists of linearly independent vectors by Lemma 3.18, the reverse inclusion holds as well in (3.14), which establishes the corollary. □

Identifying $\mathscr{L}(G)$ with $\mathfrak{g}$, Corollary 3.19 implies that the vector spaces $S(\mathfrak{g})$ and $D(G)$ are isomorphic to each other (but in general not as algebras). A particularly nice isomorphism between $S(\mathfrak{g})$ and $D(G)$ the so-called symmetrisation σ, given by the following result.

Corollary 3.20. *There exists a unique linear bijection $\sigma : S(\mathfrak{g}) \to D(G)$ such that*

$$\sigma(X^m) = \sigma(X)^m \tag{3.15}$$

for all $X \in \mathfrak{g}$ and for all non-negative integer m. Moreover if $P(X_1, \cdots, X_n) \in S(\mathfrak{g})$ and $f \in C^\infty(G)$ then we have

$$(\sigma(P)f)(g) = P(D_1, \cdots, D_n) f(g \exp(t_1 X_1 + \cdots + t_n X_n))_{\overline{t}=0}, \tag{3.16}$$

where $D_i = \frac{\partial}{\partial t_i}$.

Proof. By induction we see that the requirement (3.15) is equivalent to

$$\frac{m!}{m_1! \cdots m_n!} \sigma(X_1^{m_1} \cdots X_n^{m_n}) = X_{\overline{m}}. \tag{3.17}$$

for all tuple of non-negative integers $\overline{m}$. So defining σ by (3.17), Corollary 3.19 implies that σ sends a basis of $S(\mathfrak{g})$ to a basis of $D(G)$ and hence defines a linear isomorphism. Claim (3.16) then follows directly by (3.12). □

In particular we can define the degree of a differential operator by

$$\deg(\sigma(P)) = \deg(P)$$

for all $P \in S(\mathfrak{g})$.

3.3. **The subgroup–subalgebra correspondence.** We haven't yet properly defined the notion of a Lie subgroup. If G is a Lie group then a **Lie subgroup** of G is a submanifold of H that is

(1) a subgroup of G,
(2) and a topological group.

Remark 3.21. *Recall that by submanifold we mean immersed submanifold. In particular the topology on H may not be the subspace topology induced from G. For an example, see Remark 3.23 below.*

To check that H is indeed a Lie group, we need to check the inversion $H \to H$ and the multiplication $H \times H \to H$ are smooth maps. Since H is a topological group, these maps are continuous. On the other hand, they are the restriction of the inversion and multiplication on G, which are smooth. By Lemma 2.6 we deduce that the inversion and multiplication on H are indeed smooth maps.

Theorem 3.22. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Then given a Lie subgroup H of G, the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$. On the other hand, every subalgebra of $\mathfrak{g}$ is the Lie algebra of a unique connected Lie subgroup of G.*

Proof sketch. Let H be a Lie subgroup of G and let ι be the inclusion of H in G. By Proposition 3.8, $d\iota_{e_G}$ is a morphism of Lie algebras from $\mathfrak{h}$ to $\mathfrak{g}$. Since H is a submanifold, $d\iota_{e_G}$ is injective, and hence we might identify $\mathfrak{h}$ with its image in $\mathfrak{g}$ via $d\iota_{e_G}$. Now let $\mathfrak{h}$ be any subalgebra of $\mathfrak{g}$. Let H be the smallest subgroup of G containing $\exp(\mathfrak{h})$. To define a topology on H, we let a basis of open neighbourhoods U of e in H be defined by $U = \exp(\mathfrak{h} \cap N)$, where N is a small neighbourhood of 0 in $\mathfrak{g}$. One needs to check that this makes H a Lie subgroup of G. We omit the details, and we refer the reader to [Hel01, Chapter II, §2, Th. 2.1]. To prove that the Lie algebra of H is $\mathfrak{h}$, observe that if $X \in \mathfrak{h}$ then the map $\mathbb{R} \to H : t \mapsto \exp(tX)$ is continuous, by Lemma 2.6 it is therefore smooth, and differentiating at $t = 0$ we conclude that X belongs to the Lie algebra of H. Hence $\mathfrak{h} \subset \mathscr{L}(H)$. On the other hand we have $\dim H = \dim \mathfrak{h}$ and thus the equality is proved. Moreover H is connected since it is generated by $\exp \mathfrak{h}$ which is a connected neighbourhood of e in H. To prove the uniqueness, observe that if H_1 is a subgroup of G whose Lie algebra is $\mathfrak{h}$ then we must have $\exp(\mathfrak{h}) \subset H_1$ and thus $H \subset H_1$. Hence assuming moreover that H_1 is connected, then $H = H_1$ set-theoretically. But since the exponential is a diffeomorphism of a neighbourhood of 0 in $\mathfrak{h}$ onto a neighbourhood of e in H and H_1, we have $H = H_1$ as Lie groups. □

Remark 3.23. *This theorem is the reason why we have defined submanifolds as immersed submanifolds rather than embedded submanifolds. Indeed, consider the Lie group* $G = (\mathbb{R}/\mathbb{Z})^2$. *Its Lie algebra is* $\mathfrak{g} = \mathbb{R}^2$, *and the exponential map is given by* $(x, y) \mapsto (x \mod \mathbb{Z}, y \mod \mathbb{Z})$. *The non-trivial Lie algebras of* $\mathfrak{g}$ *are the lines* $ax + by = 0$. *The corresponding subgroup in* G *is the reduction of this line modulo* $\mathbb{Z} \times \mathbb{Z}$. *In particular if* $b \neq 0$ *and* $\frac{a}{b}$ *is irrational, this subgroup is dense in* G *and is in particular not an embedded submanifold.*

An important by-product of the proof of Theorem 3.22 is that given a Lie group G with Lie algebra $\mathfrak{g}$ and given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the unique connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$ is generated by the elements $\exp(X)$ where $X \in \mathfrak{h}$. We give a kind of converse.

Proposition 3.24. *Let* G *be a Lie group and let* H *be a Lie subgroup with respective Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$. *Assume that* H *has at most countably many connected components. Then*

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \textit{ for all } t \in \mathbb{R}\}.$$

We omit the proof and we refer to [Hel01, Chapter II, §2, Prop. 2.7]. Furthermore we have the following.

Theorem 3.25 (Cartan)**.** *Let* G *be a Lie group and let* H *be an abstract subgroup of* G *that is also a closed subset of* G. *Then* H *is a Lie subgroup of* G.

Remark 3.26. *In fact Cartan's theorem gives a more precise statement, however the above will be sufficient for us.*

Proof idea. We again only give a very rough sketch, and we refer the reader to [Hel01, Chapter II, §2, Th. 2.3] for the details. On lets

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

Using that H is closed, one shows that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Thus by Theorem 3.22 there is a unique connected subgroup $H^\star$ of G whose Lie algebra is $\mathfrak{h}$. One then shows that $H^\star$ equals the connected component of the identity in H (for the relative topology of G). □

Finally we prove the following.

Lemma 3.27. *Let* G *be a connected Lie group and let* $\phi : G \to G'$ *be a morphism of Lie groups. Then the image* $\phi(G)$ *is a Lie subgroup of* G' *with Lie algebra* $d_{e_G}\phi(\mathfrak{g}) \subset \mathfrak{g}'$.

Proof. Since G is connected, it is generated by the elements $\exp(X)$, where $X \in \mathfrak{g}$. Thus the group $\phi(G)$ is generated by the elements $\phi(\exp(X))$. On the other hand the connected subgroup of G' whose Lie algebra is $d_{e_G}\phi(\mathfrak{g})$ is generated by the elements $\exp(d_{e_G}\phi(X))$, where $X \in \mathfrak{g}$. By Lemma 3.15 we have $\phi(\exp(X)) = \exp(d_{e_G}\phi(X))$ and the result is proved. □

4. Structure of semisimple Lie groups and Lie algebras

In the sequel we shall need a robust understanding of the structure of semisimple Lie groups G, and of their algebra of differential operators. We start with (defining and) studying semisimple Lie algebras, which will allow us to derive information on semisimple Lie groups using Theorem 3.22.

4.1. **Structure of semisimple Lie algebras.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Given $X \in \mathfrak{g}$ we define an endomorphism $\mathrm{ad}_{\mathfrak{g}}(X) \in \mathrm{End}(\mathfrak{g})$ by $\mathrm{ad}_{\mathfrak{g}}(X)(Y) = [X, Y]$. When the Lie algebra $\mathfrak{g}$ is clear from the context, we shall omit it from the notation. Note that for all $X, Y, Z \in \mathfrak{g}$ we have by the Jacobi identity

$$\begin{aligned} \mathrm{ad}([Y,Z])(X) &= [[Y,Z],X] \\ &= [Y,[Z,X]] + [Z,[X,Y]] \\ &= (\mathrm{ad}(Y)\circ\mathrm{ad}(Z) - \mathrm{ad}(Z)\circ\mathrm{ad}(Y))(X) \end{aligned} \tag{4.1}$$

Moreover if σ is an automorphism of $\mathfrak{g}$ then for all $X, Y \in \mathfrak{g}$ we have

$$\mathrm{ad}(\sigma X)(Y) = [\sigma X, Y] = \sigma[X, \sigma^{-1}Y] = \sigma \circ \mathrm{ad}(X) \circ \sigma^{-1}(Y) \tag{4.2}$$

We then define a symmetric bilinear form B on $\mathfrak{g}$ by $B(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$. The bilinear form B is called the **Killing form** of $\mathfrak{g}$. We note the following properties of the Killing form. Using (4.2) and $\mathrm{tr}(\sigma \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \sigma^{-1}) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ we have

$$B(\sigma X, \sigma Y) = B(X, Y). \tag{4.3}$$

Moreover using (4.1) and $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$ we have

$$\begin{aligned} B(X,[Y,Z]) &= \mathrm{tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y)\circ\mathrm{ad}(Z) - \mathrm{ad}(X)\circ\mathrm{ad}(Z)\circ\mathrm{ad}(Y)) \\ &= B(Y,[Z,X]) = B(Z,[X,Y]). \end{aligned} \tag{4.4}$$

Example 4.1. *Consider the case of $\mathfrak{gl}_n$. It is a vector space of dimension n^2, and a basis is given by $(E_{ij})_{1\le i,j\le n}$, where E_{ij} is the matrix whose only non-zero entry is on line i and column j, and equals 1. Then we have*

$$\mathrm{ad}(X)\circ\mathrm{ad}(Y)(E_{ij}) = XYE_{ij} + E_{ij}YX - XE_{ij}Y - YE_{ij}X.$$

The (i,j)-th entry of this matrix is given by $(XY)_{ii} + (YX)_{jj} - X_{ii}Y_{jj} - Y_{ii}X_{jj}$ and summing this over $1 \le i, j \le n$ we obtain $B(X, Y) = 2n\,\mathrm{tr}(XY) - 2\,\mathrm{tr}(X)\,\mathrm{tr}(Y)$.

Remark 4.2. *Note that in general the Killing form of a Lie subalgebra* $\mathfrak{b}$ *might not be the restriction of the Killing form of* $\mathfrak{g}$ *to* $\mathfrak{b}$*. However if* $\mathfrak{b}$ *is an* **ideal** *(that is to say* $[\mathfrak{b}, \mathfrak{g}] \subset \mathfrak{b}$*) then for any* $X \in \mathfrak{b}$ *the matrix of the endomorphism* $\mathrm{ad}(X)$ *is of the form* $\begin{bmatrix} \mathrm{ad}(X)_{|\mathfrak{b}} & * \\ 0 & 0 \end{bmatrix}$ *from which follows that in this case, the Killing form of* $\mathfrak{b}$ *is indeed given by the restriction of the Killing form of* $\mathfrak{g}$*. In particular the Lie algebra* $\mathfrak{sl}_n$ *consisting of matrices whose trace is zero is an ideal of* $\mathfrak{gl}_n$ *and hence its Killing form is given by* $B(X, Y) = 2n\,\mathrm{tr}(XY)$*.*

Recall that a bilinear form B on a finite-dimensional space is degenerate if there exists a non-zero vector $X \in V$ such that $B(X, Y) = 0$ for all $Y \in X$. Otherwise B is nondegenerate. The Lie algebra $\mathfrak{g}$ is called **semisimple** if the Killing form B is nondegenerate.

Remark 4.3. *If* Z *belongs to the centre of* $\mathfrak{g}$ *then by definition* $\mathrm{ad}(Z) = 0$ *and hence* $B(Z, X) = 0$ *for all* $X \in \mathfrak{g}$*. Thus semisimple Lie algebras have trivial centre. In particular* $\mathfrak{gl}_n$ *is not semisimple (but* $\mathfrak{sl}_n$ *is since* $B(X, E_{ij}) = 2nX_{ji}$ *for all* $X \in \mathfrak{sl}_n$*).*

An automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $\theta \circ \theta = \mathrm{Id}_{\mathfrak{g}}$ is called a **Cartan involution** of $\mathfrak{g}$ if the bilinear form given by $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite on $\mathfrak{g}$.

Theorem 4.4. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then a Cartan involution for* $\mathfrak{g}$ *exists. Moreover any two Cartan involution for* $\mathfrak{g}$ *are conjugate to each other by an element of* $\mathrm{GL}(\mathfrak{g})$*.*

Proving this theorem would go beyond the scope of this course. For a proof, we refer the reader to [Wis01].

Example 4.5. *In the case of* $\mathfrak{sl}_n$ *we claim that* $\theta(X) = -{}^tX$ *is a Cartan involution. Indeed,* θ *is clearly an involutive automorphism of* $\mathfrak{sl}_n$*, and for all* $X \in \mathfrak{gl}_n \setminus \{0\}$ *we have*

$$B_\theta(X, X) = 2n \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}^2 > 0.$$

In the remainder of the course we shall only use the existence of a Cartan involution, and thus if we are willing to restrict ourselves to $\mathfrak{g} = \mathfrak{sl}_n$ *then Theorem 4.4 is not needed.*

Now let $\mathfrak{g}$ be a semisimple Lie algebra, and let θ be a Cartan involution. Then by (4.3), θ is self-adjoint with respect to the Killing form. Hence we have a B-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \tag{4.5}$$

where $\mathfrak{p}$ is the -1-eigenspace of θ and $\mathfrak{k}$ is the 1-eigenspace of θ. Note that $\mathfrak{k}$ is in particular a Lie subalgebra of $\mathfrak{g}$ (but $\mathfrak{p}$ is not). The decomposition (4.5) is called the

Cartan decomposition. Let $Z \in \mathfrak{p}$. Then by (4.4) we have

$$\begin{aligned} B_\theta(Y, [Z, X]) &= B(Y, [Z, \theta X]) \\ &= B(\theta X, [Y, Z]) = B_\theta(X, [Z, Y]) \end{aligned}$$

and thus the operator $\mathrm{ad}(Z)$ is self-adjoint with respect to B_θ. Similarly if $Z \in \mathfrak{k}$ then $\mathrm{ad}(Z)$ is an anti-symmetric operator.

Example 4.6. *For $\mathfrak{g} = \mathfrak{sl}_n$ we have seen that $\theta(X)$ is given by $-{}^tX$. Thus in this cases, $\mathfrak{p}$ consists of those matrices in $\mathfrak{g}$ that are symmetric, and $\mathfrak{k}$ is given by the anti-symmetric matrices.*

Now let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal commutative Lie subalgebra. Then the operators $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ commute with each other, as seen from (4.1). Furthermore they are self-adjoint with respect to B_θ, and in particular they are diagonalisable. We recall the following theorem from linear algebra.

Theorem 4.7. *Let V be a finite dimensional real vector space endowed with a inner product and let $\mathscr{F}$ be a family of self-adjoint endomorphisms of V that commute pairwise. Then V has an orthonormal basis consisting of vectors that are eigenvectors of every $T \in \mathscr{F}$.*

For every $\lambda \in \mathfrak{a}^*$ let

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Then λ is called a **root** if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$. If λ is a root and $v \in \mathfrak{g}_\lambda$, then v is called a **root vector**. Then by Theorem 4.7 we have a B_θ-orthogonal sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda, \tag{4.6}$$

where Σ is the set of roots.

Remark 4.8. *Let α be a root and let $X_\alpha \in \mathfrak{g}_\alpha$. Then given $H \in \mathfrak{a}$ we have $\theta([H, X_\alpha]) = [\theta(H), \theta(X_\alpha)]$. Since $H \in \mathfrak{a} \subset \mathfrak{p}$ we have $\theta(H) = -H$ and thus we obtain $\alpha(H)\theta(X_\alpha) = -[H, \theta(X_\alpha)]$, which says that $\theta(X_\alpha)$ is a root vector for $-\alpha$.*

For each root α the set $S_\alpha = \{H \in \mathfrak{a} : \alpha(H) = 0\}$ is an hyperplane in $\mathfrak{a}$. We say an element $H \in \mathfrak{a}$ is **singular** if $H \in S_\alpha$ for some $\alpha \in \Sigma$, and H is **regular** otherwise. Thus the set of regular elements in $\mathfrak{a}$ is the union of finitely many convex, open connected components, called **Weyl chambers**. We fix once and for all a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. In particular given any root $\alpha \in \Sigma$ we either have $\alpha(H) > 0$ for all $H \in \mathfrak{a}^+$ or $\alpha(H) < 0$ for all $H \in \mathfrak{a}^+$. In the first case we say α is a **positive root**, otherwise we say α is negative. A positive root is **simple** if it is not the sum of two positive roots. Let us denote by Σ^+ the set of positive roots.

Theorem 4.9 (Iwasawa decomposition). *Let* $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$*. Then* $\mathfrak{n}$ *is a subalgebra of* $\mathfrak{g}$*, and* $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$*. Moreover we have*

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

Proof. Let α, β be any two roots and let $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\beta \in \mathfrak{g}_\beta$. Let $H \in \mathfrak{a}$. Then by the Jacobi identity we have

$$\begin{aligned}[H, [X_\alpha, X_\beta]] &= -[X_\alpha, [X_\beta, H]] - [X_\beta, [H, X_\alpha]] \\ &= \beta(H)[X_\alpha, X_\beta] - \alpha(H)[X_\beta, X_\alpha] = (\alpha+\beta)(H)[X_\alpha, X_\beta].\end{aligned}$$

Thus

$$[X_\alpha, X_\beta] \in \mathfrak{g}_{\alpha+\beta}. \tag{4.7}$$

In particular if α, β are positive roots then either $\alpha + \beta$ is also a positive root or $\mathfrak{g}_{\alpha+\beta} = \{0\}$. Now using bilinearity it is easy to check that $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$. The fact that $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$ is obvious from the definition of root spaces. In view of (4.5), to prove the last statement, it suffices suffices to show that any $X \in \mathfrak{p}$ can be written in a unique way as

$$X = N + H + Z \tag{4.8}$$

with $N \in \mathfrak{n}$, $H \in \mathfrak{a}$ and $Z \in \mathfrak{k}$. From (4.6) and by Remark 4.8 we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n} \oplus \theta(\mathfrak{n}). \tag{4.9}$$

In particular any $X \in \mathfrak{p}$ can be written as $X = X_0 + N_1 + \theta(N_2)$ with $X_0 \in \mathfrak{g}_0$ and $N_1, N_2 \in \mathfrak{n}$. But then we have $-X = \theta(X) = \theta(X_0) + \theta(N_1) + N_2$, from which we deduce $N_1 = -N_2$ and $\theta(X_0) = -X_0$, that is $X_0 \in \mathfrak{p}$. On the other hand we have $[\mathfrak{a}, X_0] = 0$ and thus by maximality of $\mathfrak{a}$ we have $X_0 \in \mathfrak{a}$. Thus we can take $N = 2N_1$, $H = X_0$ and $Z = -(N_1 + \theta(N_1)) \in \mathfrak{k}$ in (4.8). To show uniqueness, we have $-X = \theta(X) = \theta(N) - H + Z$ from which follows $Z = -\frac{1}{2}(N + \theta(N))$ and hence $X = H + \frac{1}{2}N - \frac{1}{2}\theta(N)$ and since the sum (4.9) is direct we are done. □

Remark 4.10. *Let* $\mathfrak{q}$ *be the B-orthogonal complement of* $\mathfrak{a}$ *in* $\mathfrak{p}$*. Then as a by-product of the proof we have shown that the map* $N \mapsto N - \theta(N)$ *is an isomorphism between* $\mathfrak{n}$ *and* $\mathfrak{q}$ *(recall that* $\mathfrak{a}$ *and* $\mathfrak{n}$ *are orthogonal to each other).*

Example 4.11. *Consider* $\mathfrak{g} = \mathfrak{sl}_n$ *with the Cartan involution given by* $X \mapsto -{}^tX$*. Then it is easily checked that the set* $\mathfrak{a}$ *of diagonal matrices in* $\mathfrak{g}$ *is a maximal abelian subspace of* $\mathfrak{p}$*. Now if* $H \in \mathfrak{a}$ *has diagonal entries* H_i *and* $X \in \mathfrak{g}$ *then the* (i, j)*-th entry of* $[H, X] = HX - XH$ *is given by* $(H_i - H_j)X_{ij}$*. Thus we see that the roots are the linear maps* $\alpha_{ij} : H \mapsto H_i - H_j$ *where* $i \neq j$*, and* E_{ij} *is a root vector for* α_{ij}*. The regular elements in* $\mathfrak{a}$ *are those whose diagonal entries are pairwise distinct, and in particular a choice of a positive Weyl chamber is given by* $\mathfrak{a}^+ = \{H \in \mathfrak{a} : H_1 > H_2 > \cdots > H_n\}$*. Given this choice, the root* α_{ij} *is positive if and only if* $j > i$*. Thus in this case* $\mathfrak{n}$ *consists of all the strictly upper triangular matrices.*

Later on we shall need the following.

Proposition 4.12. *Any linear form on $\mathfrak{a}$ is a linear combination of simple roots.*

Proof. The Killing form is positive-definite on $\mathfrak{a} \subset \mathfrak{p}$. Thus for any linear form λ on $\mathfrak{a}$ there exists $A_\lambda \in \mathfrak{a}$ such that $\lambda(H) = B(A_\lambda, H)$ for all $H \in \mathfrak{a}$. It suffices to show that $\mathfrak{a}$ is spanned by the elements A_α, where α ranges over the simple roots, or equivalently, over all the roots. Assume $H_0 \in \mathfrak{a}$ is orthogonal to each A_α. Then for any root vector X_α we have $\operatorname{ad} H_0(X_\alpha) = 0$ and thus $\operatorname{ad} H_0$ is identically zero on $\mathfrak{g}$. But then $B_\theta(H_0, H_0) = -\operatorname{tr}(\operatorname{ad} H_0 \circ \operatorname{ad} \theta H_0) = 0$ and hence $H_0 = 0$. □

4.2. **The adjoint group.** Our aim is to use the structure theory of semisimple Lie algebra to establish analogous results at the level of Lie groups. However we need a bit of preparation. While general Lie groups are abstract manifolds, Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$ have a concrete realisation as matrix groups, which makes them easier to study. To any Lie group G we will attach such a matrix group, the adjoint group of G. Furthermore for connected Lie groups, we do not lose too much information by passing from G to its adjoint group – namely we only kill the centre of G.

Let G be any Lie group. For $g \in G$ consider the smooth map $I_G(g) : G \to G : x \mapsto gxg^{-1}$. Let us denote its differential by $\mathrm{Ad}_G(g) = d_{e_G} I_G(g)$, so $\mathrm{Ad}_G(g) \in \mathrm{GL}(\mathfrak{g})$, and in fact Ad_G is a representation of G on the vector space $\mathfrak{g}$. We call $\mathrm{Ad}_G(G)$ the **adjoint group** of G. By Lemma 3.15 we have

$$\exp(\mathrm{Ad}_G(g)X) = g \exp(X) g^{-1} \tag{4.10}$$

for all $g \in G$ and for all $X \in \mathfrak{g}$.

Remark 4.13. *Assume that G is connected. Then G is generated by the elements $\exp(X)$ where $X \in \mathfrak{g}$. Hence by (4.10) the kernel of Ad is the centre Z_G of G.*

Furthermore the map Ad_G is smooth and by differentiation we have

$$d_{e_G} \mathrm{Ad}_G = \mathrm{ad}_{\mathfrak{g}} \tag{4.11}$$

(for the details see [Hel01, Chap. II §5]). Now $\mathrm{GL}(\mathfrak{g})$ is a Lie group whose Lie algebra is $\mathrm{End}(\mathfrak{g})$, and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is a subalgebra of $\mathrm{End}(\mathfrak{g})$.

Remark 4.14. *Assume G is connected. Using Lemma 3.27 equation (4.11) implies that the Lie algebra of $\mathrm{Ad}_G(G)$ is $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$. We also note that by Lemma 3.15 the identity (4.11) above implies*

$$\mathrm{Ad}_G(\exp X) = \exp(\mathrm{ad}_{\mathfrak{g}} X) \tag{4.12}$$

for all $X \in \mathfrak{g}$.

Lemma 4.15. *Let G be a connected Lie group whose Lie algebra is commutative. Then G is commutative.*

Proof. Let $X, Y \in \mathfrak{g}$ and let $g = \exp(Y)$, $h = \exp(X)$. Then by (4.12) we have

$$\mathrm{Ad}_G(g) = \exp(\mathrm{ad}_{\mathfrak{g}} Y) = \exp(0) = \mathrm{Id}_{\mathfrak{g}}.$$

Thus by (4.10)

$$ghg^{-1} = \exp(\mathrm{Ad}_G(g)X) = \exp(X) = h.$$

Since G is generated by $\exp(\mathfrak{g})$ the result follows. □

In between $\mathrm{Ad}_G(G)$ and $\mathrm{GL}(\mathfrak{g})$ there is the group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$. It is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$ and thus by Cartan's Theorem 3.25 it is also a Lie subgroup. We now identify its Lie algebra. A **derivation** of $\mathfrak{g}$ is an endomorphism $D : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$D[X, Y] = [DX, Y] + [X, DY]$$

for all $X, Y \in \mathfrak{g}$.

Proposition 4.16. *Let $\mathfrak{g}$ be any finite-dimensional Lie algebra. The Lie algebra of* $\mathrm{Aut}(\mathfrak{g})$ *is the algebra $\partial(\mathfrak{g})$ of derivations on $\mathfrak{g}$.*

Proof. If $D \in \mathscr{L}(\mathrm{Aut}(\mathfrak{g}))$ then for all $t \in \mathbb{R}$ we have $\exp(tD) \in \mathrm{Aut}(\mathfrak{g})$, that is $\exp(tD)[X, Y] = [\exp(tD)X, \exp(tD)Y]$ for all $X, Y \in \mathfrak{g}$. Differentiating with respect to t we obtain $D[X, Y] = [DX, Y] + [X, DY]$, that is, D is a derivation. Conversely if D is any derivation then by induction we have

$$D^k[X, Y] = \sum_{i+j=k} \frac{k!}{i!j!}[D^i X, D^j y] \tag{4.13}$$

for all $X, Y \in \mathfrak{g}$ and for all natural number k. By Remark 3.14 we have

$$\exp(tD) = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k$$

for all $t \in \mathbb{R}$, and thus (4.13) implies $\exp(tD)[X, Y] = [\exp(tD)X, \exp(tD)Y]$ for all $X, Y \in \mathfrak{g}$, that is $\exp(tD) \in \mathrm{Aut}(\mathfrak{g})$ for all $t \in \mathbb{R}$. By Proposition 3.24 we conclude $D \in \partial(\mathfrak{g})$. □

To summarise, we have the following inclusions of Lie groups and Lie algebras.

$$\begin{array}{lllll} \mathrm{Ad}_G(G) & \subset & \mathrm{Aut}(\mathfrak{g}) & \subset & \mathrm{GL}(\mathfrak{g}) \\ \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) & \subset & \partial(\mathfrak{g}) & \subset & \mathrm{End}(\mathfrak{g}). \end{array}$$

Lemma 4.17. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then* $\mathrm{ad}(\mathfrak{g}) = \partial(\mathfrak{g})$.

Proof. Since $\mathfrak{g}$ is semisimple, we have $\ker(\mathrm{ad}) = \{0\}$ and thus $\mathrm{ad}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$, and in particular semisimple. Next if $D \in \partial(g)$ then for all $X, Y \in \mathfrak{g}$ we have

$$\begin{aligned}\mathrm{ad}(DX)(Y) &= [DX, Y] = D[X, Y] - [X, DY] \\ &= (D \circ \mathrm{ad}\, X)(Y) - (\mathrm{ad}\, X \circ D)(Y) = [D, \mathrm{ad}\, X](Y).\end{aligned}$$

This shows that $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\partial(\mathfrak{g})$. Let $\mathfrak{b}$ be its orthogonal complement for the Killing form B of $\partial(\mathfrak{g})$. By (4.4) if $X \in \mathfrak{b}$ then for all $Y \in \partial(\mathfrak{g})$ and for all $Z \in \mathrm{ad}(\mathfrak{g})$ we have

$$0 = B(X, [Y, Z]) = B(Z, [X, Y])$$

and thus $[X, Y] \in \mathfrak{b}$, that is $\mathfrak{b}$ is also an ideal. By Remark 4.2 $\mathfrak{b} \cap \mathrm{ad}(\mathfrak{g})$ is orthogonal to $\mathrm{ad}(\mathfrak{g})$ with respect to the Killing form on $\mathrm{ad}(\mathfrak{g})$. Since $\mathrm{ad}(\mathfrak{g})$ is semisimple it follows that $\mathfrak{b} \cap \mathrm{ad}(\mathfrak{g}) = \{0\}$. Thus if $D \in \mathfrak{b}$ and $X \in \mathfrak{g}$ then we have $\mathrm{ad}(DX) = [D, \mathrm{ad}\, X] \in \mathfrak{b} \cap \mathrm{ad}(\mathfrak{g}) = \{0\}$. Using again that $\mathfrak{g}$ is semisimple we have $DX = 0$ for all $X \in \mathfrak{g}$, that is $D = 0$. Therefore $\mathfrak{b} = \{0\}$, which implies $\mathrm{ad}(\mathfrak{g}) = \partial(\mathfrak{g})$. □

Corollary 4.18. *Let* G *be a connected Lie group with Lie algebra* $\mathfrak{g}$*. Assume* $\mathfrak{g}$ *is semisimple. Then* $\mathrm{Ad}_G(G)$ *is the connected component of the identity in* $\mathrm{Aut}(\mathfrak{g})$.

4.3. **Structure of semisimple Lie groups.** The Lie group G is **semisimple** if its Lie algebra $\mathfrak{g}$ is semisimple. Let us fix a semisimple Lie group G and let us retain notations from Section 4.1. By Theorem 3.22 there is a unique connected Lie subgroup K of G that has Lie algebra $\mathfrak{k}$.

Lemma 4.19. *Let* G *be a connected semisimple Lie group. The group* $\mathrm{Ad}_G(K)$ *is compact.*

Proof. Let $\mathrm{Aut}(\mathfrak{k}; \mathfrak{g})$ be the subgroup of $\mathrm{Aut}(\mathfrak{g})$ that preserve $\mathfrak{k}$. If $\sigma \in \mathrm{Aut}(\mathfrak{k}; \mathfrak{g})$ then by (4.3) for all $X \in \mathfrak{p}$ and for all $Y \in \mathfrak{k}$ we have $B(X, \sigma Y) = B(\sigma^{-1} X, Y) = 0$ and thus $\sigma Y \in \mathfrak{k}^\perp = \mathfrak{p}$. Thus using (4.3) again we have

$$\mathrm{Aut}(\mathfrak{k}; \mathfrak{g}) \subset O_B(\mathfrak{k}) \times O_B(\mathfrak{p}),$$

where $O_B(\mathfrak{k})$ (respectively $O_B(\mathfrak{p})$) is the group of isometries of $\mathfrak{k}$ (resp. $\mathfrak{p}$) with respect to the Killing form B. Since B is definite both on $\mathfrak{k}$ and on $\mathfrak{p}$ the groups $O_B(\mathfrak{k})$ and $O_B(\mathfrak{p})$ are compact, and so is $O_B(\mathfrak{k}) \times O_B(\mathfrak{p})$. Now $\mathrm{Aut}(\mathfrak{k}; \mathfrak{g})$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$, thus it is also closed in $O_B(\mathfrak{k}) \times O_B(\mathfrak{p})$, and hence compact. Using (4.11) and Lemma 3.27 the Lie algebra of $\mathrm{Ad}_G(K)$ is $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Let $\partial(\mathfrak{k}; \mathfrak{g})$ be the subalgebra of those derivations D of $\mathfrak{g}$ that preserve $\mathfrak{k}$. We claim that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k}) = \partial(\mathfrak{k}; \mathfrak{g})$. The inclusion $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k}) \subset \partial(\mathfrak{k}; \mathfrak{g})$ is obvious. Conversely if $D \in \partial(\mathfrak{k}; \mathfrak{g})$ then by Lemma 4.17

we have $D = \mathrm{ad}_{\mathfrak{g}}(X)$ for some $X \in \mathfrak{g}$ such that $[X, Y] \in \mathfrak{k}$ for all $Y \in \mathfrak{k}$. Write $X = X_{\mathfrak{p}} + X_{\mathfrak{k}}$ with $X_{\mathfrak{p}} \in \mathfrak{p}$ and $X_{\mathfrak{k}} \in \mathfrak{k}$. Then we have

$$[X_{\mathfrak{k}}, Y] + [X_{\mathfrak{p}}, Y] = [X, Y] = \theta[X, Y] = [\theta X, \theta Y] = [X_{\mathfrak{k}}, Y] - [X_{\mathfrak{p}}, Y]$$

from which we deduce $[X_{\mathfrak{p}}, Y] = 0$ for all $Y \in \mathfrak{k}$, and hence $D = \mathrm{ad}_{\mathfrak{g}}(X_{\mathfrak{k}})$. Next the same argument as in Proposition 4.16 shows that $\partial(\mathfrak{k}; \mathfrak{g})$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{k}; \mathfrak{g})$. Hence $\mathrm{Ad}_G(K)$ is the connected component of the identity in $\mathrm{Aut}(\mathfrak{k}; \mathfrak{g})$, and in particular it is closed in $\mathrm{Aut}(\mathfrak{k}; \mathfrak{g})$. Thus $\mathrm{Ad}_G(K)$ is compact, as claimed. □

Proposition 4.20. *Let G be a connected semisimple Lie group with finite centre. The group K defined above is compact.*

Proof. By Remark 4.13 we have $\mathrm{Ad}_G(K) \simeq K/(Z_G \cap K)$ and since Z_G is finite it follows by Lemma 4.19 that K is compact. □

Remark 4.21. *In fact, under the same assumptions it can be shown that $K = \mathrm{Ad}_G^{-1}(\mathrm{Ad}_G(K))$, which implies in particular that $Z_G \subset K$. This follows from the Cartan decomposition (Theorem 4.25 below), but we will not prove this here.*

Example 4.22. *Let $G = \mathrm{SL}_2(\mathbb{R})$. Then $\mathfrak{g} = \mathfrak{sl}_2 = \langle e_1, e_2, e_3 \rangle$ where $e_1 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $e_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $e_3 = \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$. Then $\mathfrak{k} = \mathbb{R}e_1$ and $K = \mathrm{SO}_2(\mathbb{R})$. Let $k = \left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right] \in K$. Then we have*

$$\begin{aligned} k(I_2 + he_1)k^{-1} &= I_2 + he_1, \\ k(I_2 + he_2)k^{-1} &= I_2 + h\left[\begin{smallmatrix} 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \\ \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \end{smallmatrix}\right], \\ k(I_2 + he_3)k^{-1} &= I_2 + h\left[\begin{smallmatrix} \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{smallmatrix}\right], \end{aligned}$$

from which follows that the matrix of $\mathrm{Ad}_G(k)$ in the basis (e_1, e_2, e_3) is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 0 & -2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2\theta) & \sin(2\theta) \\ 0 & -\sin(2\theta) & \cos(2\theta) \end{bmatrix}.$$

So the restriction of Ad_G to K may be thought of as the map $\mathrm{SO}_2(\mathbb{R}) \to SO_2(\mathbb{R}) : k \mapsto k^2$. Its kernel is $\{I_2, -I_2\} = Z_{\mathrm{SL}_2(\mathbb{R})}$.

4.3.1. *The Iwasawa decomposition.* In order to deduce the Iwasawa decomposition at the level of the Lie group from the Iwasawa decomposition of its Lie algebra we first need to following lemma.

Lemma 4.23. *Let G be any Lie group and let $\mathfrak{g}$ be its Lie algebra. Suppose $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$ where $\mathfrak{s}$ and $\mathfrak{h}$ are Lie subalgebras. Let S and H be the connected Lie subgroups of G with Lie algebras $\mathfrak{s}$ and $\mathfrak{h}$ respectively. Then the map*

$$m : S \times H \to G : (s, h) \mapsto sh$$

is regular.

Proof. Given $s \in S$ and $h \in H$ we have
$$T_s S = d_e \ell_s \mathfrak{s}, \quad T_h H = d_e \ell_h \mathfrak{h}, \quad T_{sh} G = d_e \ell_{sh} \mathfrak{g}$$
Furthermore we may identify
$$T_{(s,h)}(S \times H) \cong T_s S \oplus T_h H.$$
Now given $X \in \mathfrak{s}$ and $Y \in \mathfrak{h}$ we have
$$m(s \exp(tX), h) = sh \exp(t \operatorname{Ad}(h^{-1} Y)),$$
$$m(s, h \exp(tY)) = sh \exp(tY),$$
from which follows that
$$d_{(s,h)} m(d_e \ell_s X, d_e \ell_h Y) = d_e \ell_{sh}(\operatorname{Ad}(h^{-1}) X + Y).$$
Now if $\operatorname{Ad}(h^{-1})X + Y = 0$ then $X + \operatorname{Ad}(h)Y = 0$ and since $\operatorname{Ad}(h)Y \in \mathfrak{h}$ it follows that $X = Y = 0$. □

Theorem 4.24 (Iwasawa decomposition)**.** *Let G be a connected semisimple Lie group with finite centre. Let $\mathfrak{g}$ be the Lie algebra of G and let*
$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$$
be an Iwasawa decomposition. Let N, A, K be the connected subgroups of G with Lie algebras $\mathfrak{n}, \mathfrak{a}, \mathfrak{k}$ respectively. The group A normalises N. Moreover the map
$$N \times A \times K \to G$$
$$(n, a, k) \mapsto nak$$
is a diffeomorphism from $N \times A \times K$ onto G.

Proof sketch. The proof proceeds in two steps: one first proves the theorem for the adjoint group $\operatorname{Ad}_G(G)$, then one "lifts" the result to the group G. We only present a sketch of the first step. For more details, we refer the reader to [Hel62, Chap. VI, § 5, Thm 5.1] or [Kna02, Theorem 6.46]. Let $\bar{G} = \operatorname{Ad}_G(G)$, $\bar{N} = \operatorname{Ad}_G(N)$, $\bar{A} = \operatorname{Ad}_G(A)$, $\bar{K} = \operatorname{Ad}_G(K)$. We recall that $\bar{G}$ is a closed subgroup of $\operatorname{GL}(\mathfrak{g})$ First, we argue that $\bar{A}$ normalises $\bar{N}$. To show this, let $H \in \operatorname{ad}_{\mathfrak{g}}(\mathfrak{a})$ and $X \in \operatorname{ad}_{\mathfrak{g}}(\mathfrak{n})$. Let $a = \exp(H)$. By (4.10) we have
$$a^{-1}(\exp X)a = \exp(\operatorname{Ad}_G(a)X).$$
Now by (4.12) we have
$$\operatorname{Ad}_G(a)(X) = \exp(\operatorname{ad}_{\mathfrak{g}} H)(X).$$

Now since $[\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a}), \mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})] \subset \mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})$ we have $(\mathrm{ad}_{\mathfrak{g}} H)^k(X) \in \mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})$ for all integer k and thus by Remark 3.14 it follows that $\mathrm{Ad}_G(a)(X) \in \mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})$. Hence

$$a^{-1}\exp(X)a \in \exp(\mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})) \subset \bar{N}$$

as desired. It particular, it follows that $\bar{N}\bar{A}$ is a subgroup of G. Next we argue $\bar{N}\bar{A}$ is a closed subgroup. Pick a basis $\mathscr{B}$ of $\mathfrak{g}$ consisting of root vectors and of elements of $\mathfrak{g}_0$. Then for $H \in \mathfrak{a}$ the matrix of $\mathrm{ad}_{\mathfrak{g}} H$ in the basis $\mathscr{B}$ is diagonal. By Remark 3.14 it easily follows that $\bar{A}$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$ and hence of $\bar{G}$. Similarly, using that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta},$$

we may order $\mathscr{B}$ in such a way that for all $X \in \mathfrak{n}$ the matrix of X is upper triangular with zeroes on the diagonal. Now letting T be the Lie algebra of all upper triangular matrices with zeroes on the diagonal, one can check that $\exp(T)$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$ (consisting of upper triangular matrices with ones on the diagonal), and thus $\bar{N} = \bar{G} \cap \exp(T)$ is closed in $\bar{G}$. To deduce $\bar{N}\bar{A}$ is closed, let $g = \lim_m n_m a_m$ with $a_m \in \bar{A}$ and $n_m \in \bar{N}$. Let a be the diagonal matrix with the same diagonal entries as g. Then $\lim_m a_m = a$ and since $\bar{A}$ being closed, we have $a \in \bar{A}$. Moreover $ga^{-1} = \lim_m n_m \in \bar{N}$ since $\bar{N}$ is closed. Since $\bar{N}\bar{A}$ is a closed subgroup of $\bar{G}$, we conclude by Cartan's Theorem 3.25 that $\bar{N}\bar{A}$ is a Lie subgroup of $\bar{G}$. In particular the multiplication map

$$m_1 : \bar{N} \times \bar{A} \to \bar{N}\bar{A}$$

is smooth and onto. The Lie algebra of $\bar{N}\bar{A}$ is $\mathfrak{n} \oplus \mathfrak{a}$ and thus by Lemma 4.23 the map m_1 is also regular. It is also one-to-one as one may recover a and n from the diagonal entries of na. Thus m_1 is a diffeomorphism. Next the multiplication map

$$m_2 : \bar{N}\bar{A} \times \bar{K} \to \bar{G}$$

is smooth. Its image is closed because $\bar{K}$ is compact and $\bar{N}\bar{A}$ is closed. Using Lemma 4.23 again, m_2 is everywhere regular, and since

$$\dim(\bar{G}) = \dim(\mathfrak{n}) + \dim(\mathfrak{a}) + \dim(\mathfrak{k}) = \dim(\bar{N}\bar{A}) + \dim(\bar{K}),$$

the image of m_2 is also open in $\bar{G}$. Since $\bar{G}$ is connected, this shows that m_2 is onto. To conclude the first step, it remains to show that m_2 is one-to-one, that is $\bar{K} \cap \bar{N}\bar{A} = \{1\}$. To this end, one can for instance use the fact that if $g \in \bar{N}\bar{A}$ and $g \neq 1$ then the subgroup generated by g is not compact. □

By Theorem 4.24 each $g \in G$ can be written in a unique way as $g = nak$ with $n \in N, a \in A$ and $k \in K$. Moreover Lemma 4.15 the group A is commutative. Hence we know by Corollary 3.17 that the exponential $\exp : \mathfrak{a} \to A$ is a surjective group homomorphism. We claim it is also injective. Indeed, assume $H \in \mathfrak{a}$ is such that $\exp(H) = e_G$. Then by (4.12) we have

$$e_{\mathrm{Ad}(G)} = \mathrm{Ad}_G(\exp(H)) = \exp(\mathrm{ad}_{\mathfrak{g}} H).$$

Now it follows from Remark 3.14 and the properties of the matrix exponential that $\mathrm{ad}_{\mathfrak{g}} H = 0$ and since $\mathfrak{g}$ is semisimple, we must have $H = 0$ as claimed. Henceforth given $g \in G$ we define $A(g) \in \mathfrak{a}$ as the unique element such that $g \in N \exp(A(g))K$. When $g = a \in A$ we may occasionally write $\log a$ for $A(g)$.

4.3.2. *The Cartan decomposition.* In previous subsection we "lifted" the Iwasawa decomposition at the level of the Lie algebra to deduce the Iwasawa decomposition at the level of the Lie group. Similarly, we now "lift" the Cartan decomposition (4.5) at the Lie group level.

Theorem 4.25 (Cartan decomposition, version 1)**.** *Let G be a connected semisimple group with finite centre. Let $\mathfrak{g}$ be the Lie algebra of G and let*

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$$

be a Cartan decomposition. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}$. Then the map

$$\begin{aligned} \mathfrak{p} \times K &\to G \\ (X, k) &\mapsto \exp(X)k \end{aligned}$$

is a diffeomorphism from $\mathfrak{p} \times K$ onto G.

We omit the proof. As in the case of the Iwasawa decomposition, one first proves it for the adjoint group, for which one can use the matrix structure, then one "lifts" the theorem to G itself. We defer the reader to [Kna02, Theorem 6.31].

Lemma 4.26. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and let $\mathfrak{a}$ be a maximal commutative subalgebra of $\mathfrak{p}$. Let $H \in \mathfrak{a}$ be a regular element. Then we have*

$$Z_{\mathfrak{g}}(H) = \{X \in \mathfrak{p} : [H, X] = 0\} = \mathfrak{g}_0.$$

Proof. Let $X \in \mathfrak{p}$ such that $[H, X] = 0$. Write $X = X_0 + \sum_{\lambda \in \Sigma} X_\lambda$ with $X_0 \in \mathfrak{g}_0$ and $X_\lambda \in \mathfrak{g}_\lambda$ for each $\lambda \in \Sigma$. Then we have

$$0 = [H, X_0] + \sum_{\lambda \in \Sigma} \lambda(H) X_\lambda.$$

Since by assumption $\lambda(H) \neq 0$ for each $\lambda \in \Sigma$, the directness of the sum (4.6) implies $X_\lambda = 0$ for each $\lambda \in \Sigma$, that is $X = X_0 \in \mathfrak{g}_0$. ∎

Lemma 4.27. *Let G be a connected semisimple group with finite centre. Let $\mathfrak{g}$ be the Lie algebra of G, let*

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$$

be a Cartan decomposition, and let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}$. Then we have

$$\mathrm{Ad}_G(K)\mathfrak{a} = \mathfrak{p}.$$

Proof. Let $H \in \mathfrak{a}$ be a regular element (this is possible because Σ is finite, and for $\alpha \in \Sigma$ the kernel $\ker(\alpha)$ is an hyperplane in $\mathfrak{a}$) and let $X \in \mathfrak{p}$. Consider the continuous map

$$\begin{aligned} K &\to \mathbb{R} \\ k &\mapsto B(H, \mathrm{Ad}_G(k)X). \end{aligned}$$

By compactness, it admits a minimum in some $k_0 \in K$. Now let $Z \in \mathfrak{k}$. Then the map

$$\begin{aligned} \mathbb{R} &\to \mathbb{R} \\ t &\mapsto B(H, \mathrm{Ad}_G(\exp tZ)\, \mathrm{Ad}_G(k_0)X). \end{aligned}$$

is smooth and admits a minimum at $t = 0$. Differentiating with respect to t we obtain

$$\begin{aligned} 0 &= \left(\frac{d}{dt} B(H, \mathrm{Ad}_G(\exp tZ)\, \mathrm{Ad}_G(k_0)X)\right)_{t=0} \\ &= B(H, (\mathrm{ad}_{\mathfrak{g}} Z)\, \mathrm{Ad}_G(k_0)X) \\ &= B(H, [Z, \mathrm{Ad}_G(k_0)X]) = B(Z, [\mathrm{Ad}_G(k_0)X, H]). \end{aligned}$$

Using (4.12), (4.2) and the fact that the exponential on $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is given by a power series, one shows that $\theta(\mathrm{Ad}_G(k_0)X) = \mathrm{Ad}_G(k_0)(\theta X)$ and thus $\mathrm{Ad}_G(k_0)X \in \mathfrak{p}$. This implies in particular $[\mathrm{Ad}_G(k_0)X, H] \in \mathfrak{k}$. Since $Z \in \mathfrak{k}$ is arbitrary and B is negative-definite on $\mathfrak{k}$, it follows that $[\mathrm{Ad}_G(k_0)X, H] = 0$ and thus by Lemma 4.26 we have $\mathrm{Ad}_G(k_0)X \in \mathfrak{p} \cap \mathfrak{g}_0 = \mathfrak{a}$, which establishes the result. □

Corollary 4.28. *Let G be a connected semisimple group with finite centre. Let $\mathfrak{g}$ be the Lie algebra of G, let*

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$$

be a Cartan decomposition, and let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Let A, K be the connected subgroups of G with Lie algebra $\mathfrak{a}, \mathfrak{k}$ respectively. Then we have

$$G = KAK.$$

Proof. This follows from combining Theorem 4.25 and Lemma 4.27, remembering that by (4.10) we have

$$\exp(\mathrm{Ad}_G(k)H) = k\exp(H)k^{-1}$$

for all $k \in K$ and $H \in \mathfrak{a}$, and that $\exp : \mathfrak{a} \to A$ is surjective by Corollary 3.17. □

In contrast with the Iwasawa decomposition and the statement of Theorem 4.25, the decomposition $g = k_1 a k_2$ (with $k_1, k_2 \in K$ and $a \in A$) is not unique in general. In order to account for this, we need to introduce the Weyl group.

4.3.3. *The Weyl group.* The Weyl group is a group of permutation of the roots generated by certain reflections. It can also be realised as a quotient of certain subgroups of G. We now start setting up the stage. In this section we fix a connected semisimple group G with finite centre, we let $\mathfrak{g}$ be its Lie algebra and we retain the notations from Sections 4.1 and 4.3. Since the Killing form B is non-degenerate, given $\lambda \in \mathfrak{g}^*$ there exists $A_\lambda \in \mathfrak{g}$ such that

$$\lambda(H) = B(A_\lambda, H)$$

for all $H \in \mathfrak{g}$. In this way, we may define a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}^*$ by setting for all $\lambda, \mu \in \mathfrak{g}^*$

$$\langle \lambda, \mu \rangle = B(A_\lambda, A_\mu) = \lambda(A_\mu) = \mu(A_\lambda).$$

We identify $\mathfrak{a}^*$ as the subspace of $\mathfrak{g}^*$ consisting of those linear forms that vanish on $\mathfrak{a}^\perp$, or equivalently as the subspace consisting of those linear forms λ such that $A_\lambda \in \mathfrak{a}$. The bilinear form $\langle \cdot, \cdot \rangle$ is in particular positive-definite on $\mathfrak{a}^*$. Then for $\alpha \in \mathfrak{a}^* \setminus \{0\}$ we may define a reflection s_α on $\mathfrak{a}^*$ by

$$s_\alpha(\lambda) = \lambda - 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

Note that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\lambda) = \lambda$ for $\lambda \in \alpha^\perp$. Similarly we can transfer the adjoint action of G on $\mathfrak{g}$ to an action of G on $\mathfrak{g}^*$ by defining for all $\lambda \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$

$$(\mathrm{Ad}^*_G(g)\lambda)(X) = \lambda(\mathrm{Ad}_G(g^{-1})X).$$

This is compatible with the assignment $\lambda \mapsto A_\lambda$ in the sense that the linear form $\mathrm{Ad}^*_G(g)\lambda$ is represented by $\mathrm{Ad}_G(g)A_\lambda$ (this can be shown taking the exponential and using (4.4) and (4.12)). Given a root α we want to show that

- if β is a root, then so is $s_\alpha(\beta)$,
- there exists $k \in K$ such that $\mathrm{Ad}^*_G(k) = s_\alpha$.

Lemma 4.29. *Let $\alpha \in \Sigma$ and let $X_\alpha \in \mathfrak{g}_\alpha$, normalised so that $B(X_\alpha, \theta(X_\alpha)) = -1$. Then we have*

$$[X_\alpha, \theta(X_\alpha)] = -A_\alpha.$$

Moreover let $E_\alpha = \frac{\pi}{\sqrt{2\langle\alpha,\alpha\rangle}} X_\alpha$ and $k = \exp(E_\alpha + \theta E_\alpha) \in K$. Then

$$\mathrm{Ad}_G(k)\mathfrak{a} = \mathfrak{a}.$$

and

$$\mathrm{Ad}_G^*(k) = s_\alpha.$$

Proof. Recall that $\theta(X_\alpha)$ is a root vector with root $-\alpha$. Next we observe that $[X_\alpha, \theta(X_\alpha)] \in \mathfrak{a}$. Indeed, it is immediate that $[X_\alpha, \theta(X_\alpha)] \in \mathfrak{p}$, and $[X_\alpha, \theta(X_\alpha)]$ commutes with $\mathfrak{a}$ since for all $H \in \mathfrak{a}$ we have

$$\begin{aligned}[H, [X_\alpha, \theta(X_\alpha)]] &= -[X_\alpha, [\theta(X_\alpha), H]] - [\theta(X_\alpha), [H, X_\alpha]] \\ &= -\alpha(H)([X_\alpha, \theta(X_\alpha)] + [\theta(X_\alpha), X_\alpha]) = 0.\end{aligned}$$

Finally we have

$$B(H, [X_\alpha, \theta(X_\alpha)]) = B(X_\alpha, [\theta(X_\alpha), H]) = B(X_\alpha, \alpha(H)\theta(X_\alpha)) = -\alpha(H),$$

which proves the first claim. Next let $H \in \mathfrak{a}$ such that $\alpha(H) = 0$. Then $\mathrm{ad}_\mathfrak{g}(E_\alpha)(H) = \mathrm{ad}_\mathfrak{g}(\theta E_\alpha)(H) = 0$ from which, using (4.12) and Remark 3.14 it follows that

$$\begin{aligned}\mathrm{Ad}_G(k)H &= \exp(\mathrm{ad}_\mathfrak{g}(E_\alpha + \theta E_\alpha))H \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}_\mathfrak{g}(E_\alpha + \theta E_\alpha))^n H = H.\end{aligned} \tag{4.14}$$

On the other hand since $A_\alpha \in \mathfrak{a}$ we have

$$\mathrm{ad}_\mathfrak{g}(X_\alpha)A_\alpha = [X_\alpha, A_\alpha] = -\alpha(A_\alpha)X_\alpha = -\langle\alpha,\alpha\rangle X_\alpha$$

and similarly

$$\mathrm{ad}_\mathfrak{g}(\theta X_\alpha)A_\alpha = \langle\alpha,\alpha\rangle\theta(X_\alpha).$$

Thus

$$\mathrm{ad}_\mathfrak{g}(X_\alpha + \theta X_\alpha)A_\alpha = \langle\alpha,\alpha\rangle(\theta X_\alpha - X_\alpha).$$

Applying $\mathrm{ad}_\mathfrak{g}(X_\alpha + \theta X_\alpha)$ to this identity and using the first part we deduce

$$\mathrm{ad}_\mathfrak{g}(X_\alpha + \theta X_\alpha)^2 A_\alpha = -2\langle\alpha,\alpha\rangle A_\alpha.$$

Thus
(4.15)

$$
\begin{aligned}
\mathrm{Ad}_G(k)A_\alpha &= \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}_{\mathfrak{g}}(E_\alpha + \theta E_\alpha))^n A_\alpha \\
&= \sum_{m=0}^{\infty} \frac{1}{(2m)!}(\mathrm{ad}_{\mathfrak{g}}(E_\alpha + \theta E_\alpha))^{2m} A_\alpha + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}(\mathrm{ad}_{\mathfrak{g}}(E_\alpha + \theta E_\alpha))^{2m+1} A_\alpha \\
&= \sum_{m=0}^{\infty} \frac{1}{(2m)!}(-\pi^2)^m A_\alpha + \pi\sqrt{\frac{\langle \alpha, \alpha\rangle}{2}} \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}(-\pi^2)^m (\theta X_\alpha - X_\alpha) \\
&= (\cos\pi)A_\alpha + \pi\sqrt{\frac{\langle \alpha, \alpha\rangle}{2}}(\sin\pi)(\theta X_\alpha - X_\alpha) = -A_\alpha.
\end{aligned}
$$

Combining (4.14) and (4.15) shows that $\mathrm{Ad}_G(k)$ indeed preserves $\mathfrak{a}$, and that the restriction of $\mathrm{Ad}_G^*(k)$ to $\mathfrak{a}^*$ agrees with s_α. □

Corollary 4.30. *Let $\alpha, \beta \in \Sigma$. Then $s_\alpha(\beta) \in \Sigma$.*

Proof. We are to show that $\mathfrak{g}_{s_\alpha(\beta)} \neq \{0\}$. Let k be as in the statement of Lemma 4.29. Let $H \in \mathfrak{a}$ and let $X \in \mathfrak{g}_\beta$. Then we have

$$
\begin{aligned}
[H, \mathrm{Ad}_G(k)X] &= \mathrm{Ad}_G(k)[\mathrm{Ad}_G(k)^{-1}H, X] = \mathrm{Ad}_G(k)\big(\beta(\mathrm{Ad}_G(k)^{-1}H)X\big) \\
&= \mathrm{Ad}_G(k)\big((\mathrm{Ad}_G^*(k)\beta)(H)X\big) \\
&= \mathrm{Ad}_G(k)\big(s_\alpha(\beta)(H)X\big) = s_\alpha(\beta)(H)(\mathrm{Ad}_G(k)X),
\end{aligned}
$$

and thus $\mathrm{Ad}_G(k)X$ is a root vector for $s_\alpha(\beta)$. □

Definition 4.31. We define the **Weyl group** W of G as the finite subgroup of $\mathrm{GL}(\mathfrak{a}^*)$ spanned by the reflections s_α, where α ranges in Σ.

Corollary 4.30 implies that the Weyl group permutes the roots. Moreover since the roots span $\mathfrak{a}^*$, an element of the Weyl group is completely determined by the corresponding permutation, hence we may identify the Weyl group with its image in the group of all permutations of Σ. In particular the Weyl group is finite. Just as we used the identification of $\mathfrak{g}^*$ to $\mathfrak{g}$ to transport the adjoint action of G to an action on $\mathfrak{g}^*$, we obtain in the reverse direction an action of the Weyl group on $\mathfrak{a}$ by setting

$$s \cdot A_\lambda = A_{s(\lambda)}$$

for all $s \in W$ and $\lambda \in \mathfrak{a}^*$. The second statement of Lemma 4.29 directly extends in the sense that for any $s \in W$ there exists $k \in K$ such that $\mathrm{Ad}_G(k)\mathfrak{a} = \mathfrak{a}$ and $\mathrm{Ad}_G^*(k) = s$. Since moreover the form $s(\lambda) = \mathrm{Ad}_G^*(k)\lambda$ is represented by $\mathrm{Ad}_G(k)A_\lambda$, this element k satisfies $s \cdot H = \mathrm{Ad}_G(k)H$ for all $H \in \mathfrak{a}$. Equivalently, given $H \in \mathfrak{a}$ we may define $s \cdot H$ by

$$\beta(s \cdot H) = (s^{-1}\beta)(H) \tag{4.16}$$

for all $\beta \in \mathfrak{a}^*$. To see this let $k \in K$ as above and observe that

$$\beta(s \cdot H) = B(A_\beta, \mathrm{Ad}_G(k)H) = B(\mathrm{Ad}_G(k)^{-1}A_\beta, H) = (s^{-1}\beta)(H).$$

Example 4.32. *We have seen in Example 4.11 that the roots of $\mathfrak{sl}_n$ are the maps α_{ij} : $H \mapsto H_i - H_j$, where H_i is the i-th diagonal entry of $H \in \mathfrak{a}$. Thus the reflections $s_{\alpha_{ij}}$ acts on the root $\alpha_{\ell m}$ by*

$$s_{\alpha_{ij}}(\alpha_{\ell m}) = \alpha_{\tau(\ell),\tau(m)},$$

where $\tau \in \mathfrak{S}_n$ is the transposition (ij). Since $\mathfrak{S}_n$ is generated by the transpositions, we may identify the Weyl group of $\mathfrak{sl}_n$ with $\mathfrak{S}_n$. Given a root $\alpha = \alpha_{ij}$ the vector $A_\alpha \in \mathfrak{a}$ is the diagonal matrix whose only non-zero entries are the i-th entry and the j-th entry, given respectively by $\frac{1}{2n}$ and $-\frac{1}{2n}$. Hence the action of $\sigma \in W \simeq \Sigma_n$ on any $H \in \mathfrak{a}$ is given by the diagonal matrix $\sigma \cdot H$ whose i-th digonal entry is $H_{\sigma(i)}$. On the other hand given a permutation $\sigma \in \Sigma_n$ let P_σ be the matrix whose i-th row only has zeroes except on column $\sigma(i)$ where it has a one. Then given $H \in \mathfrak{a}$ the i-th diagonal entry of $P_\sigma H P_\sigma^{-1}$ is given by $H_{\sigma(i)}$, which says that

$$P_\sigma H P_\sigma^{-1} = \sigma \cdot H.$$

Moreover if σ has signature 1 then $P_\sigma \in \mathrm{SO}_n(\mathbb{R})$ and hence an element k corresponding to σ can be taken to be P_σ. On the other hand if σ has signature -1 then changing the sign of a single entry of P_σ we obtain an element of $\mathrm{SO}_n(\mathbb{R})$ satisfying the desired property.

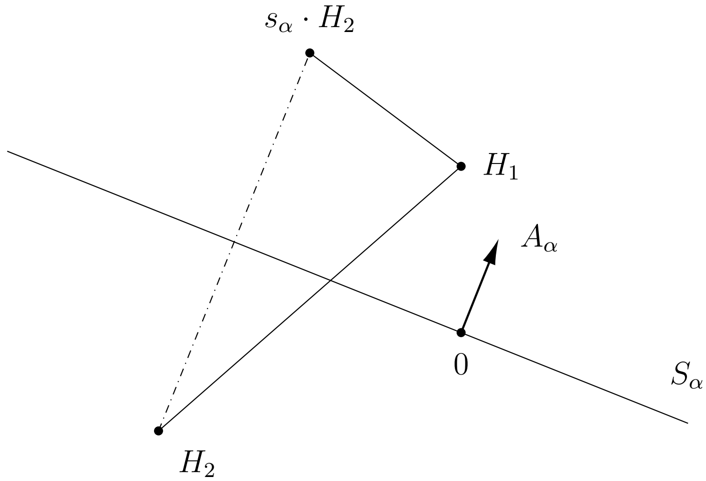
We now study in more details the action of the Weyl group. We need the following lemma.

Proposition 4.33. *The Weyl group permutes transitively the set of Weyl chambers. Moreover for any $H \in \mathfrak{a}$ the orbit $W \cdot H$ intersects the closure $\overline{\mathfrak{a}}^+$ of $\mathfrak{a}^+$ in exactly one point.*

Proof. We have seen already that W permutes the roots α. Thus (4.16) implies that W permutes the hyperplanes $S_\alpha = \ker(\alpha)$, and hence the Weyl chambers. To see that this action is transitive, let H_1, H_2 lying in two distinct Weyl chambers $\mathfrak{a}^{(1)}$ and $\mathfrak{a}^{(2)}$ respectively. Then the segment $[\![H_1, H_2]\!]$ intersects the hyperplane S_α for some $\alpha \in \Sigma$. Now we claim

$$\|H_1 - s_\alpha \cdot H_2\| < \|H_1 - H_2\|,$$

where the norm is induced from the Killing form.



Indeed since by definition S_α is the orthogonal of A_α we may write $H_1 = S_1 + t_1 A_\alpha$ and $H_2 = S_2 - t_2 A_\alpha$ with $S_1, S_2 \in S_\alpha$ and $t_1, t_2 \in \mathbb{R}$ with the same sign. Then $s_\alpha \cdot H_2 = S_2 + t_2 A_2$ and thus by Pythagoras theorem

$$\begin{aligned}\|H_1 - s_\alpha \cdot H_2\|^2 &= \|S_1 - S_2\| + (t_1 - t_2)^2 \|A_\alpha\|^2 \\ &< \|S_1 - S_2\| + (t_1 + t_2)^2 \|A_\alpha\|^2 = \|H_1 - H_2\|^2.\end{aligned}$$

Now as s varies over the finite group W the distance $\|H_1 - s \cdot H_2\|$ reaches a strict minimum for $s = s_0$ say. Then the segment $[\![H_1, s_0 \cdot H_2]\!]$ does not intersect any hyperplane S_α, thus H_1 and $s_0 \cdot H_2$ lie in the same Weyl chamber. Thus $s_0 \cdot \mathfrak{a}^{(2)} = \mathfrak{a}^{(1)}$ and thus W acts transitively on the set of Weyl chambers. To prove the second claim, let $H_1, H_2 \in \mathfrak{a}^+$. Then we have seen that for all $s \in W$ we have $\|H_1 - s \cdot H_2\| \geq \|H_1 - H_2\|$, which implies

$$B(H_1, H_2) \geq B(H_1, s \cdot H_2). \tag{4.17}$$

By continuity this inequality still holds for all $H_1, H_2 \in \overline{\mathfrak{a}}^+$. Now if $s_0 \cdot H_2 \in \overline{\mathfrak{a}}^+$ then replacing H_2 with $s_0 H_2$ and applying (4.17) with $s = s_0^{-1}$ we deduce

$$B(H_1, H_2) = B(H_1, s_0 \cdot H_2)$$

and hence

$$B(H_1, H_2 - s_0 \cdot H_2) = 0$$

for all $H_1 \in \overline{\mathfrak{a}}^+$. This implies $H_2 = s_0 \cdot H_2$ and the result is proved. □

Theorem 4.34. *Let*

$$N_K(\mathfrak{a}) = \{k \in K : \mathrm{Ad}_G(k)\mathfrak{a} = \mathfrak{a}\}$$

and

$$Z_K(\mathfrak{a}) = \{k \in K : \mathrm{Ad}_G(k)H = H \text{ for all } H \in \mathfrak{a}\}.$$

Then there is a group isomorphism

$$\begin{aligned} W &\to N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \\ s &\mapsto k_s \end{aligned}$$

such that for all $s \in W$ we have

$$s \cdot H = \mathrm{Ad}_G(k_s)H \text{ for all } H \in \mathfrak{a}. \tag{4.18}$$

Moreover W permutes simply transitively the set of Weyl chambers.

Proof. We have already seen that given $s \in W$ there exists $k \in N_K(\mathfrak{a})$ such that (4.18) holds for k. Clearly (4.18) holds for k' too if and only if $k' \in kZ_K(\mathfrak{a})$ and so we get an injective map $W \to N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ with the desired property. For the surjectivity, by Proposition 4.33 it suffices to show that the adjoint action of $N_K(\mathfrak{a})$ permutes the roots and that if $k \in N_K(\mathfrak{a})$ is such that $\mathrm{Ad}_G(k)\mathfrak{a}^+ = \mathfrak{a}^+$ then $k \in Z_K(\mathfrak{a})$; that is the action of $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is simply transitive on the set of Weyl chambers. Indeed, assuming this is the case, and let $k \in N_K(\mathfrak{a})$ and let $\mathfrak{a}^{(1)} = \mathrm{Ad}_G(k)\mathfrak{a}^+$. By Proposition 4.33 there exists $s \in W$ such that $s \cdot \mathfrak{a}^+ = \mathfrak{a}^{(1)}$, hence by (4.18) we have $\mathfrak{a}^{(1)} = \mathrm{Ad}_G(k_s)\mathfrak{a}^+$. Now by uniqueness of k we must have $k \in k_sZ_K(\mathfrak{a})$. For the proof that the action of $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is indeed simply transitive on the set of Weyl chambers, we refer the reader to [Kna02, Theorem 6.57] or [Hel01, Chap. VII Theorem 2.12]. □

Lemma 4.35. *Let $H \in \mathfrak{a}$ and let $k \in K$ such that*

$$\mathrm{Ad}_G(k)H \in \mathfrak{a}.$$

Then there exists $s \in W$ such that

$$\mathrm{Ad}_G(k)H = s \cdot H.$$

Proof. Fix $X \in Z_{\mathfrak{g}}(H) \cap \mathfrak{p}$ and $H_1 \in \mathfrak{a}^+$. By the same compactness argument as in the proof of Lemma 4.27 we find $z \in Z_G(H) \cap K$ such that

$$B([H_1, \mathrm{Ad}_G(z)X], T) = 0$$

for all $T \in Z_{\mathfrak{g}}(H) \cap \mathfrak{k}$. As in Lemma 4.27 we have $[H_1, \mathrm{Ad}_G(z)X] \in \mathfrak{k}$. Moreover by the Jacobi identity

$$\begin{aligned} [H, [H_1, \mathrm{Ad}_G(z)X]] &= -[H_1, [\mathrm{Ad}_G(z)X, H]] - [\mathrm{Ad}_G(z)X, [H, H_1]] \\ &= -[H_1, \mathrm{Ad}_G(z)[X, \mathrm{Ad}_G(z^{-1})H]] \\ &= -[H_1, \mathrm{Ad}_G(z)[X, H]] = 0, \end{aligned}$$

thus $[H_1, \mathrm{Ad}_G(z)X] \in Z_{\mathfrak{g}}(H) \cap \mathfrak{k}$. As T is arbitrary we deduce $[H_1, \mathrm{Ad}_G(z)X] = 0$ and since H_1 is regular by Lemma 4.26 we have

$$X \in \mathrm{Ad}_G(z^{-1})\mathfrak{a}.$$

Observe that if $H_2 \in \mathfrak{a}$ then we have

$$[\mathrm{Ad}_G(k^{-1})H_2, H] = \mathrm{Ad}_G(k^{-1})[H_2, \mathrm{Ad}_G(k)H] = 0$$

and in particular by Lemma 4.27 we may take $X = \mathrm{Ad}_G(k^{-1})H_2$. Thus taking $H_2 \in \mathfrak{a}$ arbitrary we deduce

$$\mathrm{Ad}_G(k^{-1})\mathfrak{a} \subset \mathrm{Ad}_G(z^{-1})\mathfrak{a}$$

and since both side have the same dimension, they are equal. Hence $kz^{-1} \in N_K(\mathfrak{a})$ and by Theorem 4.34 there is $s \in W$ such that

$$s \cdot H = \mathrm{Ad}_G(kz^{-1})H = \mathrm{Ad}_G(k)H.$$

□

Theorem 4.36 (Cartan decomposition, version 2)**.** *Keep the notations as above and let* $\overline{A^+} = \exp(\overline{\mathfrak{a}}^+)$. *Then*

$$G = K\overline{A^+}K,$$

and if

$$k_1 a k_2 = k_1' a' k_2'$$

with $k_1, k_2, k_1', k_2' \in K$ *and* $a, a' \in \overline{A^+}$ *then* $a = a'$.

Proof. For the existence, let $g \in G$. By Corollary 4.28 and the surjectivity of $\exp : \mathfrak{a} \to A$ there exist $k_1, k_2 \in K$ and $H \in \mathfrak{a}$ such that

$$g = k_1 \exp(H) k_2.$$

Now by Proposition 4.33 there exists $s \in W$ and $H^+ \in \overline{\mathfrak{a}}^+$ such that $H = s \cdot H^+$. Moreover as discussed above, there exists $k \in K$ such that $s \cdot H^+ = \mathrm{Ad}_G(k)H^+$. Finally using (4.10) we have

$$\exp(\mathrm{Ad}_G(k)H^+) = k \exp(H^+) k^{-1},$$

and thus

$$g = k_1 k \exp(H^+) k^{-1} k_2.$$

For the uniqueness assume $k_1 a k_2 = a'$ where $a = \exp(H_1)$ and $a' = \exp(H_2)$ with $H_1, H_2 \in \overline{\mathfrak{a}}^+$. Then $k_1 a k_1^{-1}(k_1 k_2) = a'$. Now by Lemma 4.27 and using (4.10) again, $k_1 a k_1^{-1} = \exp(\mathrm{Ad}_G(k_1)H_1) \in \exp(\mathfrak{p})$ and thus by the first version of the Cartan decomposition Theorem 4.25 we have $\mathrm{Ad}_G(k_1)H_1 = H_2$. By Lemma 4.35 there is $s \in W$ such that $H_2 = s \cdot H_1$ and moreover since we are assuming $H_1, H_2 \in \overline{\mathfrak{a}}^+$, Proposition 4.33 implies $H_1 = H_2$, as desired. □

Remark 4.37. *We can reformulate the content of this subsection as follows. There is an algebraically defined Weyl group $W(\Sigma)$ and an analytically defined Weyl group $W(K, \mathfrak{a})$. The algebraic Weyl group $W(\Sigma)$ is defined as the group generated by the reflections s_α, where $\alpha \in \Sigma$, and the analytic Weyl group $W(K, \mathfrak{a})$ is defined as the quotient $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. We have shown that these two notions coincide, and we have used it to derive the second version of the Cartan decomposition. The crucial missing ingredient in our exposition is that the action of the analytic Weyl group $W(K, \mathfrak{a})$ on the set of Weyl chambers is simply transitive.*

5. Integration on Lie groups

5.1. **Integration on manifolds.** Recall that if if $U \subset \mathbb{R}^n$ is an open set and $\varphi : U \to \mathbb{R}^n$ is smooth, its **Jacobian** $\operatorname{Jac}_{\mathbf{x}} \varphi$ at $\mathbf{x} = (x_1, \cdots, x_n) \in U$ is by definition

$$\operatorname{Jac}_{\mathbf{x}} \varphi = \det\left(d_{\mathbf{x}}\varphi\right).$$

We recall the change of variable formula for $\mathbb{R}^n$. Assume φ as above is injective and its Jacobian does not vanish on U. Then for any continuous function f with compact support included in $\varphi(U)$ we have

$$\int_{\varphi(U)} f(\mathbf{y})\, d\mathbf{y} = \int_U f(\varphi(\mathbf{x}))|\operatorname{Jac}_{\mathbf{x}} \varphi|\, d\mathbf{x}. \tag{5.1}$$

Let $\mathscr{M}$ be a smooth manifold of dimension n. For each $p \in \mathscr{M}$ let $\Omega_p \in \operatorname{Hom}(T_p\mathscr{M}, \mathbb{R}^n)$. We assume moreover that the map $\Omega : p \mapsto \Omega_p$ is smooth in the sense that for any vector field X on $\mathscr{M}$ the map $\mathscr{M} \to \mathbb{R}^n, p \mapsto \Omega_p(X_p)$ is smooth. Then for all $p \in \mathscr{M}$ and for all $\ell \in \operatorname{Hom}(\mathbb{R}^n, T_p\mathscr{M})$ we define

$$\omega_p(\ell) = \det(\Omega_p \circ \ell).$$

for all $\ell \in \operatorname{Hom}(T_p\mathscr{M}, \mathbb{R}^n)$. A map $p \mapsto \omega_p$ as above is called a n**-form** on $\mathscr{M}$.

Remark 5.1. *This definition might appear slightly different from the usual definition of a n-form, but it is equivalent.*

Remark 5.2. *Specifying an isomorphism $\ell : \mathbb{R}^n \to T_p\mathscr{M}$ is the same as specifying a basis of $T_p\mathscr{M}$. Once a basis of $T_p\mathscr{M}$ is fixed, specifying a homomorphism $\Omega_p : T_p\mathscr{M} \to \mathbb{R}^n$ amounts to specifying an ordered n-tuple of elements of $\mathbb{R}^n$. Finally taking the determinant of n vectors in $\mathbb{R}^n$ gives the (signed) volume of the (oriented) parallelepiped defined by these vectors.*

Let $p \in \mathscr{M}$ and let (U, φ) and (V, ψ) be two local charts containing p. Recall that upon identifying $\mathbb{R}^n$ to $T_{\varphi(p)}\mathbb{R}^n$, we have for all $p \in U$ a map $d_p\varphi : T_p\mathscr{M} \to \mathbb{R}^n$, and similarly for ψ. Then using the chain rule as well as the identity $d_{\varphi(p)}\varphi^{-1} = (d_p\varphi)^{-1}$ we have the following identity

$$\omega_p(d_{\varphi(p)}\varphi^{-1}) \operatorname{Jac}_{\psi(p)}(\varphi \circ \psi^{-1}) = \omega_p(d_{\psi(p)}\psi^{-1}). \tag{5.2}$$

We say two local charts (U,φ) and (V,ψ) on $\mathscr{M}$ have the same orientation if the mapping $\psi \circ \varphi^{-1}$ has positive Jacobian on $\varphi(U \cap V)$. We say that $\mathscr{M}$ is **orientable** if $\mathscr{M}$ can be covered by a collection of local charts which all have the same orientation. Once we have fixed an orientable manifold $\mathscr{M}$ as well as such a collection $\mathscr{C}$ of local charts, we say $\mathscr{M}$ is **oriented**. Given any local chart (U,φ) on $\mathscr{M}$ we say (U,φ) is positively oriented if (U,φ) has the same orientation as every local chart in $\mathscr{C}$. We shall denote by $C_c(\mathscr{M})$ the space of continuous, compactly supported functions on $\mathscr{M}$. We wish to define the integration of $f \in C_c(\mathscr{M})$ against n-forms on $\mathscr{M}$. We first assume there is a positively oriented local chart (U,φ) such that $\operatorname{Supp}(f) \subset U$. Then given a n-form ω on $\mathscr{M}$ we define

$$\int_{\mathscr{M}} f\omega = \int_{\varphi(U)} (f \circ \varphi^{-1})(\mathbf{y})\omega_{\varphi^{-1}(\mathbf{y})}(d_{\mathbf{y}}\varphi^{-1})\, d\mathbf{y}.$$

We claim that this definition is independent of the choice of positively oriented local chart containing $\operatorname{Supp}(f)$. Indeed if (V,ψ) is another such local chart then by the change of variables formula (5.1) for $\mathbb{R}^n$ with $G = \varphi \circ \psi^{-1}$ and (5.2) with $p = \psi^{-1}(\mathbf{x})$ we have

$$\int_{\varphi(U)} (f \circ \varphi^{-1})(\mathbf{y})\omega_{\varphi^{-1}(\mathbf{y})}(d_{\mathbf{y}}\varphi^{-1})\, d\mathbf{y} = \int_{\psi(V)} (f \circ \psi^{-1})(\mathbf{x})\omega_{\psi^{-1}(\mathbf{x})}(d_{\mathbf{x}}\psi^{-1})\, d\mathbf{x}.$$

Next, given any $f \in C_c(\mathscr{M})$ there exists a covering of $\mathscr{M}$ by positively oriented local charts $(U_i,\varphi_i)_{i\in I}$ and finitely many compactly supported functions f_j such that $f = \sum_j f_j$ and for each j we have $\operatorname{Supp}(f_j) \subset U_i$ for some i (this follows from a smooth partition of unity, see [Hel01, Chap. I, § 1, Thm 1.3]). We then define

$$\int_{\mathscr{M}} f\omega = \sum_j \int_{\mathscr{M}} f_j\omega.$$

Here again, we need to check that the definition does not depend on the choice of the decomposition $f = \sum_j f_j$, but we leave it as an exercise for the reader (or we refer them to [Hel00, Chap. I, § 1]).

Let $\mathscr{M}$ and $\mathscr{N}$ be oriented manifolds of dimension n and let Φ be a diffeomorphism from $\mathscr{M}$ onto $\mathscr{N}$. We say Φ is orientation-preserving if given any positively oriented local chart (U,φ) on $\mathscr{M}$, the local chart $(\Phi(U),\varphi\circ\Phi^{-1})$ on $\mathscr{N}$ is positively oriented. Let ω be a n-form on $\mathscr{N}$. Then we define the pullback $\Phi^*\omega$ of ω by

$$\left(\Phi^*\omega\right)_p(\ell) = \omega_{\Phi(p)}(d_p\Phi\circ\ell)$$

for all $\ell\in\mathrm{Hom}(\mathbb{R}^n,T_p\mathscr{M})$. Then we have the following change of variable formula

$$\int_{\mathscr{M}} f\Phi^*\omega = \int_{\mathscr{N}}(f\circ\Phi^{-1})\omega \tag{5.3}$$

for all $f\in C_c(\mathscr{M})$. It suffices to check it when the support of f is included in some local chart (U,φ) on $\mathscr{M}$. In this case choosing the local chart $(V,\psi)=(\Phi(U),\varphi\circ\Phi^{-1})$ on $\mathscr{N}$ we have $f\circ\varphi^{-1}=(f\circ\Phi^{-1})\circ\psi^{-1}$ and given $\mathbf{x}\in\varphi(U)$ we have $(\Phi^*\omega)_{\varphi^{-1}(\mathbf{x})}(d_{\mathbf{x}}\varphi^{-1}) = \omega_{\psi^{-1}(\mathbf{x})}(d_{\mathbf{x}}\psi^{-1})$ and thus both sides of (5.3) are the same integral.

5.2. **Integration on Lie groups.** Now let G be a Lie group of dimension n and let $\mathfrak{g}$ be its Lie algebra. A n-form ω on G is **left-invariant** if it satisfies $\ell_g^*\omega=\omega$ for all $g\in G$.

Lemma 5.3. *There exists a non-zero left-invariant n-form ω on G. Moreover any left-invariant n-form on G is of the form $\lambda\omega$ for some $\lambda\in\mathbb{R}$.*

Proof. Fix an isomorphism $\Omega_e\in\mathrm{Hom}(\mathfrak{g},\mathbb{R}^n)$, and for all $g\in G$ define $\Omega_g=\Omega_e\circ(d_e\ell_g)^{-1}\in\mathrm{Hom}(T_pG,\mathbb{R}^n)$. Let ω be the n-form associated to Ω. It is clearly non-zero, and we claim it is left-invariant. Indeed if $x,g\in G$ then by definition for all $f\in\mathrm{Hom}(\mathbb{R}^n,T_xG)$ we have

$$\begin{aligned}\left(\ell_g^*\omega\right)_x(f) &= \omega_{gx}(d_x\ell_g\circ f)\\ &= \det\left(\Omega_e\circ(d_e\ell_{gx})^{-1}\circ d_x\ell_g\circ f\right)\\ &= \det\left(\Omega_e\circ(d_e\ell_x)^{-1}\circ f\right) = \omega_x(f).\end{aligned}$$

Now since $d_e\ell_g$ is an isomorphism for all $g\in G$, any left-invariant n-form ω' is completely determined by ω'_e. But if ω'_e is represented by $\Omega'_e\in\mathrm{Hom}(\mathfrak{g},\mathbb{R}^n)$ then we have $\omega'_e=\lambda\omega_e$, where $\lambda=\det(\Omega'_e\circ\Omega_e^{-1})$. □

Now fix a local chart (U,φ) where U is an open connected neighbourhood of e.

Lemma 5.4. *The local charts $(gU,\varphi\circ\ell_g^{-1})$, where g ranges over G, define an orientation of G.*

Proof. It suffices to show that if $U\cap gU\neq\varnothing$ then for all $\mathbf{x}\in\varphi(U\cap g^{-1}U)$ we have $\mathrm{Jac}_{\mathbf{x}}(\varphi\circ\psi^{-1})>0$, where $\psi=\varphi\circ\ell_g^{-1}$. Let $x=\varphi^{-1}(\mathbf{x})\in U\cap g^{-1}U$. Let ω be a non-zero left-invariant n-form on G. Then by (5.2) and by left-invariance of ω we have

$$\begin{aligned}\omega_{gx}(d_{\varphi(gx)}\varphi^{-1})\,\mathrm{Jac}_{\mathbf{x}}(\varphi\circ\psi^{-1}) &= \omega_{gx}(d_{\mathbf{x}}\psi^{-1})\\ &= \omega_{gx}(d_x\ell_g\circ d_{\mathbf{x}}\varphi^{-1})\\ &= \omega_x(d_{\mathbf{x}}\varphi^{-1}).\end{aligned}$$

But the map $U\to\mathbb{R}: x\mapsto \omega_x(d_{\varphi(x)}\varphi^{-1})$ is continuous and doesn't vanish, hence it does not change sign. □

By Lemmas 5.3 and 5.4 we may define the integral of any $f\in C_c(G)$ against a non-zero left-invariant n-form ω. Moreover we may pick ω such that $\int_G f\omega\geq 0$ whenever f only takes non-negative values (by left-invariance this is equivalent to require that for our fixed local chart (U,φ) we have $\det(\Omega_e\circ d_{\varphi(e)}\varphi^{-1})>0)$. We call such a n-form a **Haar measure** and we denote $\int_G f\omega$ by $\int_G f(x)\,dx$. We may on occasion use both notations. By (5.3) we have

$$\int_G f(gx)\,dx=\int_G f(x)\,dx$$

for all $g\in G$.

We have defined the left-invariant Haar measure, but we might as well have defined the analogous notion of a right-invariant Haar measure, by requiring it to be right-invariant instead of left-invariant. The two notions are related as follows.

Lemma 5.5. *Let G be any Lie group and let dg be a left-invariant Haar measure on G. Then*

$$d^{(r)}g=\det\left(\mathrm{Ad}_G(g)\right)dg$$

is a right-invariant n-form on G.

Proof. Let ω be the left-invariant n-form and for $g\in G$ let $\omega_g^{(r)}=\det\left(\mathrm{Ad}_G(g)\right)\omega_g$. We need to check that

$$(r_x^*\omega^{(r)})_g=\omega_g^{(r)} \tag{5.4}$$

for all $x,g\in G$. To this end, it is sufficient to check that

$$(r_x^*\omega^{(r)})_e=\omega_e^{(r)}$$

for all $x\in G$. Now that $\mathrm{Ad}_G(x)=d_eI_G(x)$, where $I_G(x)=r_{x^{-1}}\circ\ell_x$. Then using the left-invariance of ω we have for all $x\in G$

$$(r_x^*\omega^{(r)})_e=\det\left(\mathrm{Ad}_G(x)\right)\left(I_G(x^{-1})^*\omega\right)$$

and evaluating both side on any $\ell\in\mathrm{Hom}(\mathbb{R}^n,\mathfrak{g})$ we obtain (5.4). □

As a consequence we have the following change of variable formula for all $f \in C_c(G)$ and for all $g \in G$

$$\int_G f(xg)\, dx = \int_G f(x) \left|\det \mathrm{Ad}_G(g)\right| \, dx. \tag{5.5}$$

To see this write $f(xg) = \frac{f(xg)}{\det \mathrm{Ad}_G(x)} \det \mathrm{Ad}_G(x)$ and use the right-invariance of $\det \mathrm{Ad}_G(x) dx$ to change variables $xg \mapsto x$. Note that the absolute value is there because r_g might not be orientation-preserving. A Lie group G is called **unimodular** if the left-invariant Haar measure is also right-invariant, which by the above discussion is equivalent to $|\det \mathrm{Ad}_G(g)| = 1$ for all $g \in G$.

Lemma 5.6. *Let G be a connected semisimple Lie group with Iwasawa decomposition $G = NAK$. Then the groups G, N, A, K are unimodular.*

Proof. The group A is abelian and in particular $\mathrm{Ad}_A = \mathrm{Id}_{\mathfrak{a}}$. We know by Lemma 4.19 that the group $\mathrm{Ad}_K(K)$ is compact. Since the map $K \to \mathbb{R}^\times : k \mapsto \det(\mathrm{Ad}_K(k))$ is a continuous group homomorphism, its image is a compact subgroup of $\mathbb{R}^\times$. Thus for all $k \in K$ we have $|\det(\mathrm{Ad}_K(k))| = 1$. As discussed in the proof of Theorem 4.24, we may pick a basis of $\mathfrak{n}$ for which the matrix of $\mathrm{Ad}_N(n)$ is triangular with ones on the diagonal for all $n \in N$, and in particular its determinant is 1. Finally $\mathrm{Ad}_G(g)$ leaves the Killing form invariant for each $g \in G$. Since the Killing form is non-degenerate it follows $(\det \mathrm{Ad}_G(g))^2 = 1$. □

Lemma 5.7. *Let G be any Lie group and let $\mathfrak{g}$ be its Lie algebra. Suppose $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$ where $\mathfrak{s}$ and $\mathfrak{h}$ are Lie subalgebras. Let S and H be the connected Lie subgroups of G with Lie algebras $\mathfrak{s}$ and $\mathfrak{h}$ respectively. Suppose*

$$m : S \times H \to G : (s, h) \mapsto sh$$

is a bijection. Then the Haar measures on S, H, G may be normalised so that

$$\int_G f(g) dg = \int_{S\times H} f(sh) \frac{\det \mathrm{Ad}_H(h)}{\det \mathrm{Ad}_G(h)} \, ds dh$$

for all $f \in C_c(G)$.

Proof. Given $s \in S$ and $h \in H$ we have

$$T_s S = d_e\ell_s \mathfrak{s}, \quad T_h H = d_e\ell_h \mathfrak{h}, \quad T_{sh} G = d_e\ell_{sh} \mathfrak{g}$$

and we may identify

$$T_{(s,h)}(S \times H) \cong T_s S \oplus T_h H. \tag{5.6}$$

Furthermore we may identify

$$
\begin{aligned}
T_sS \oplus T_hH &\cong T_{sh}G \\
d_e\ell_s X + d_e\ell_h Y &\mapsto d_e\ell_{sh}(X+Y).
\end{aligned}
\tag{5.7}
$$

Under these identification we may see a linear map $\ell : \mathbb{R}^n \to T_{(s,h)}(S\times H)$ as an element of $\tilde{\ell} \in \mathrm{Hom}(\mathbb{R}^n, T_{sh}G)$. If ω is the left-invariant on G then the n-form $\tilde{\omega}$ on $S \times H$ defined by

$$\tilde{\omega}_{(s,h)}(\ell) = \omega_{sh}(\tilde{\ell})$$

is left-invariant. Next, by Lemma 4.23, m is a diffeomorphism, and its differential is given by

$$d_{(s,h)}m(d_e\ell_s X, d_e\ell_h Y) = d_e\ell_{sh}(\mathrm{Ad}_G(h^{-1})X + Y).$$

for all $X \in \mathfrak{s}$ and $Y \in \mathfrak{h}$. Hence by (5.3) we have

$$\int_{M\times N} f(sh) m^*\omega = \int_G f(g)dg.$$

Now by definition $m^*\omega$ is the n-form on $S \times H$ given by

$$(m^*\omega)_{(s,h)}(\ell) = \omega_{sh}(d_{(s,h)}m \circ \ell)$$

for all linear map $\ell : \mathbb{R}^n \to T_{(s,h)}(S\times H)$. Using (5.6) and (5.7) to view $d_{(s,h)}m$ as a linear map from $T_{sh}G$ to itself and using the definition of n-forms, we obtain

$$(m^*\omega)_{(s,h)}(\ell) = \det(d_{(s,h)}m)\omega_{sh}(\tilde{\ell}) = \tilde{\omega}_{(s,h)}(\ell).$$

Finally

$$\mathrm{Ad}_G(h^{-1})X + Y = \mathrm{Ad}_G(h^{-1})(X + \mathrm{Ad}_H(h)Y)$$

and thus

$$\det(d_{(s,h)}m) = \frac{\det \mathrm{Ad}_H(h)}{\det \mathrm{Ad}_G(h)}.$$

□

5.3. **Integration formula for the Iwasawa decomposition.** From now on we fix a connected semisimple Lie group G with finite centre. We let $\mathfrak{g}$ be its Lie algebra and

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$$

be the Iwasawa decomposition of $\mathfrak{g}$ as in § 4.1. We let N, A, K be the connected Lie subgroups of G with Lie algebras given respectively by $\mathfrak{n}, \mathfrak{a}, \mathfrak{k}$. We have seen in Theorem 4.24 that G has Iwasawa decomposition $G = NAK$. We fix a positive Weyl chamber $\mathfrak{a}^+$ and we let Σ^+ be the set of positive roots. For any $\alpha \in \Sigma$ define the **multiplicity** $m_\alpha = \dim \mathfrak{g}_\alpha$. We define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

Theorem 5.8. *The Haar measures on G, N, A, K can be normalised so that for all $f \in C_c(G)$ we have*

$$\int_G f(g)\, dg = \int_N \int_A \int_K e^{-2\rho(\log(a))} f(nak)\, dkdadn.$$

Proof. We apply Lemma 5.7 twice: once for $N \times A \to NA$ and once for $NA \times K \to G$. Thus we may normalise the Haar measures so that

$$\int_G f(g)\, dg = \int_N \int_A \int_K f(nak) \frac{\det \mathrm{Ad}_A(a)}{\det \mathrm{Ad}_{NA}(a)} \frac{\det \mathrm{Ad}_K(k)}{\det \mathrm{Ad}_G(k)}\, dkdadn.$$

We have seen in Lemma 5.6 that $|\det \mathrm{Ad}_A(a)| = |\det \mathrm{Ad}_K(k)| = |\det \mathrm{Ad}_G(k)| = 1$. To calculate $\det \mathrm{Ad}_{NA}(a)$ pick a basis of $\mathfrak{n} + \mathfrak{a}$ consisting of elements of $\mathfrak{a}$ and of positive root vectors. Now if $X \in \mathfrak{g}_\alpha$ for some root α then we have

$$\begin{aligned} \mathrm{Ad}_{NA}(a)X &= \exp(\mathrm{ad}_{\mathfrak{n}+\mathfrak{a}}(\log a))X \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}_{\mathfrak{n}+\mathfrak{a}}(\log a))^n X \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \alpha(\log a) X \\ &= e^{\alpha(\log a)} X, \end{aligned}$$

and thus we obtain $\det \mathrm{Ad}_{NA}(a) = e^{2\rho(\log a)}$. □

Corollary 5.9. *Let the Haar measure be normalised as in Theorem 5.8. Then for all $f \in C_c(G)$ we have*

$$\int_G f(g)\, dg = \int_{A \times N \times K} f(ank)\, dadndk = \int_{K \times N \times A} f(kna)\, dkdnda.$$

Proof. For the identity we change variables $na \mapsto an$ and we use formula (5.5) together with the calculation of $\det \mathrm{Ad}_{NA}(a)$ above. We then apply the first equality to the function $g \mapsto f(g^{-1})$ to deduce the second identity. Note that the change of variable $g \mapsto g^{-1}$ takes the left-invariant Haar measures to right-invariant n-forms, however by Lemma 5.6 these are the same for G, N, A, K (possibly up to sign). □

5.4. **Harish-Chandra's integral functional equation.** We keep the set-up and notations as above.

Lemma 5.10. *The exponential map* $\mathfrak{n} \to N$ *is surjective.*

Proof. We already know by Proposition 3.16 that there are an open neighbourhoods U of 0 in $\mathfrak{n}$ and V of e in N such that the exponential is a diffeomorphism from U to V. Now let $n \in N$. We are to show that $n \in \exp(\mathfrak{n})$. Since N is the subgroup generated by $\exp(\mathfrak{n})$ there exist an integer $m \geq 1$ and $X_1, \cdots, X_m \in \mathfrak{n}$ such that $n = \exp(X_1) \cdots \exp(X_m)$. Moreover there exists an open neighbourhood $V_m \subset V$ of e in N such that the product of any m elements of V_m belongs to V. Let $U_m \subset U$ be such that $V_m = \exp(U_m)$. Now fix $H_0 \in \mathfrak{a}^+$ and given $t \in \mathbb{R}$ define $a_t = \exp(-tH_0)$. We claim that for any $X \in \mathfrak{n}$ there exists $t_0 \in \mathbb{R}$ such that for all $t \geq t_0$ we have $\mathrm{Ad}(a_t)X \in U_m$. Indeed, write $X = \sum_{\alpha \in \Sigma^+}$ with $X_\alpha \in \mathfrak{g}_\alpha$. Then

$$\mathrm{Ad}(a_t)X = \sum_{\alpha \in \Sigma^+} e^{-t\alpha(H_0)} X_\alpha \tag{5.8}$$

and since $\alpha(H_0) > 0$ for all $\alpha \in \Sigma_+$ the claim follows. In particular there exists $t \in \mathbb{R}$ such that $\mathrm{Ad}(a_t)X_k \in V_m$ for $1 \leq k \leq m$, and hence $\exp(\mathrm{Ad}(a_t)X_k) \in U_m$ Then we have

$$\begin{aligned} a_t n a_t^{-1} &= a_t \exp(X_1) a_t^{-1} \cdots a_t \exp(X_m) a_t^{-1} \\ &= \exp(\mathrm{Ad}(a_t)X_1) \cdots \exp(\mathrm{Ad}(a_t)X_m) \in U. \end{aligned}$$

Thus there exists $X \in V$ such that $a_t n a_t^{-1} = \exp(X)$. But then $n = a_t^{-1} \exp(X) a_t = \exp(\mathrm{Ad}(a_t^{-1})X)$. ☐

Lemma 5.11. *Let* $H \in \mathfrak{a}$ *be a regular element and let* $h = \exp H$*. Then the map*

$$\begin{aligned} \xi_h : N &\to N \\ n &\mapsto h^{-1} n h n^{-1} \end{aligned}$$

is a surjective diffeomorphism.

Proof. It is clear that ξ_h is smooth, and we know that A normalises N thus ξ_h takes values in N as claimed. Next we compute its differential. Let $n \in N$ and let $X \in \mathfrak{n}$. Then for all $t \in \mathbb{R}$ we have

$$\begin{aligned} \xi_h(n \exp tX) &= h^{-1} n (\exp tX) h \exp(-tX) n^{-1} \\ &= h^{-1} n h n^{-1} \exp(t \,\mathrm{Ad}(nh^{-1})X) \exp(-t \,\mathrm{Ad}(n) tX). \end{aligned}$$

Differentiating with respect to t we obtain

$$\begin{aligned} d_n\xi_h(d_e\ell_n X) &= d_e\ell_{h^{-1}nhn^{-1}}\big((\mathrm{Ad}(nh^{-1}) - \mathrm{Ad}(n))X\big) \\ &= d_e\ell_{h^{-1}nhn^{-1}}\big(\mathrm{Ad}(n)(\mathrm{Ad}(h^{-1}) - 1)X\big). \end{aligned} \tag{5.9}$$

Now write $X = \sum_{\alpha\in\Sigma^+} X_\alpha$ with $X_\alpha \in \mathfrak{g}_\alpha$. Then

$$\mathrm{Ad}(h^{-1})X = \sum_{\alpha\in\Sigma^+} e^{-\alpha(H)} X_\alpha$$

and since H is regular we have $\mathrm{Ad}(h^{-1})X \neq X$. Thus $d_n\xi_h$ does not vanish on $\mathfrak{n}$, and hence ξ_h is regular. In particular there are open neighbourhoods U and V of e in N such that ξ_h is a diffeomorphism from U to V. Moreover we may pick U and V such that $U = \exp(U_0)$ and $V = \exp(V_0)$ where U_0 and V_0 are neighbourhoods of 0 in $\mathfrak{n}$. Now let $H_0 \in \mathfrak{a}^+$ and for $t \in \mathbb{R}$ define $a_t = \exp(-tH_0)$. Observe that

$$\xi_h(a_t n a_t^{-1}) = a_t \xi_h(n) a_t^{-1}. \tag{5.10}$$

By (5.8) it is clear that

$$\bigcup_{t\in\mathbb{R}} \mathrm{Ad}(a_t)V_0 = \mathfrak{n},$$

and hence by Lemma 5.10 we have

$$\bigcup_{t\in\mathbb{R}} a_t V a_t^{-1} = N.$$

Since $V = \xi_h(U)$, equation (5.10) now implies that $\xi_h(N) = N$. Finally we need to show ξ_h is injective. So assume $\xi_h(n_1) = \xi_h(n_2)$. Then using (5.10) we have

$$\xi_h(a_t n_1 a_t^{-1}) = \xi_h(a_t n_2 a_t^{-1}).$$

As in the proof of Lemma 5.10 we may choose t so large that $a_t n_1 a_t^{-1}$ and $a_t n_2 a_t^{-1}$ both belong to U, and since ξ_h is injective on U we obtain $a_t n_1 a_t^{-1} = a_t n_2 a_t^{-1}$, which implies $n_1 = n_2$. □

Corollary 5.12. *Let $H \in \mathfrak{a}$ be a regular element and let $h = \exp(H)$. Then for all $f \in C_c(N)$ we have*

$$\int_N f(n)\,dn = \prod_{\alpha\in\Sigma^+} |1 - e^{-\alpha(H)}|^{m_\alpha} \int_N f(h^{-1}nhn^{-1})\,dn.$$

Proof. We apply the change of variable formula (5.3). Using (5.9) we have

$$\xi^*(dn) = \det\big(\mathrm{Ad}(n)(\mathrm{Ad}(h^{-1}) - 1)\big)\,dn.$$

We have seen in Lemma 5.6 that $\det \mathrm{Ad}(n) = 1$. Next the same argument as in proof of Theorem 5.8 shows that

$$\det\bigl(\mathrm{Ad}(h^{-1}) - 1\bigr) = \prod_{\alpha \in \Sigma^+} (e^{-\alpha(H)} - 1)^{m_\alpha}.$$

Note that the absolute values are there because ξ_h might not be orientation-preserving. □

Let $u \in N_K(A) = \{g \in K : gAg^{-1} = A\}$. For $(k, n) \in K \times N$ define $\phi_u(k, n) = (k_1, n_1) \in K \times N$ such that

$$uknu^{-1} \in k_1 n_1 A.$$

The map ϕ_u is well-defined and smooth by the Iwasawa decomposition. Moreover it is injective because if $\phi_u(k, n) = \phi_u(k', n')$ then by definition there is $a \in A$ such that $uknu^{-1} = uk'n'u^{-1}a$, but then $kn = k'n'u^{-1}au$ and since $u^{-1}au \in A$, by uniqueness of the Iwasawa decomposition we have $(k', n') = (k, n)$. Finally it is surjective because it has a right inverse, given by $\phi_{u^{-1}}$. Since $\phi_{u^{-1}}$ is smooth for the same reason, ϕ_u is a diffeomorphism.

Lemma 5.13. *For all $u \in N_G(A)$ we have*

$$\phi_u^*(dkdn) = dkdn.$$

Proof. For $f \in C_c(G)$ define $\overline{f}$ by

$$\overline{f}(k, n) = \int_A f(kna)\, da.$$

Since the Iwasawa decomposition is a diffeomorphism, $\overline{f}$ has compact support. On the other hand define $f^u(g) = f(ugu^{-1})$. Then we have

$$\begin{aligned} \overline{f^u}(k, n) &= \int_A f(uknau^{-1})\, da \\ &= \int_A f(uknu^{-1}uau^{-1})\, da \\ &= \int_A f(uknu^{-1}a)\, da = \overline{f} \circ \phi_u(k, n). \end{aligned}$$

On the other hand by Corollary 5.9 we have

$$\int_{K \times N} \overline{f}(k, n)\, dkdn = \int_G f(g)dg.$$

Since G is unimodular by Lemma 5.6 we have

$$\int_G f(g)dg = \int_G f^u(g)dg = \int_{K\times N} \overline{f^u}(k,n)\,dkdn,$$

and thus we obtain

$$\int_{K\times N} \overline{f}(k,n)\,dkdn = \int_{K\times N} \overline{f}\circ\phi_u(k,n)\,dkdn.$$

Now the result follows from (5.3) provided that the map $C_c(G) \to C_c(K \times N) : f \mapsto \overline{f}$ is surjective, which we prove in the next lemma. □

Lemma 5.14. *Given $f \in C_c^\infty(G)$ define*

$$\overline{f}(k,n) = \int_A f(kna)\,da.$$

for all $(k,n) \in K \times N$. Then the map $C_c^\infty(G) \to C_c^\infty(K \times N) : f \mapsto \overline{f}$ is surjective.

Proof. Let $F \in C_c^\infty(K \times N)$ with compact support $C \subset K \times N$. Assume $C \neq \varnothing$ otherwise there is nothing to prove. Let $C_A \subset A$ be any compact subset of positive measure. Let $\tilde{C} = \{kna : (k,n) \in C, a \in C_A\}$. Then $\tilde{C}$ is a compact subset of G. Now let $f_1 \in C_c(G)$ be non-negative and such that

$$\tilde{C} \subset \operatorname{Supp}(f_1).$$

Then $\overline{f_1}$ is positive on C and the function f defined by

$$f(g) = \begin{cases} F(k,n)\frac{f_1(g)}{\overline{f_1}(k,n)} & \text{if } g \in knA \text{ with } (k,n) \in K \times N, \\ 0 & \text{otherwise} \end{cases}$$

is continuous and satisfies $\overline{f}(k,n) = F(k,n)$ for all $(k,n) \in K \times N$. □

Theorem 5.15 (Harish-Chandra)**.** *Assume G has finite centre. For $s \in W$ and $H \in \mathfrak{a}$ denote $a = \exp(H)$ and $a^s = \exp(s \cdot H)$. Let $f \in C_c(G)$ satisfy $f(kgk^{-1}) = f(g)$ for all $k \in K$ and $g \in G$. Then the function defined by*

$$F_f(a) = e^{\rho(\log(a))} \int_N f(an)\,dn$$

satisfies the functional equation $F_f(a^s) = F_f(a)$ for all $s \in W$.

Proof. Normalise the Haar measures as in Theorem 5.8 and so that $\int_K dk = 1$. First assume H is regular. By Corollary 5.12 we have

$$\begin{aligned} F_f(a) &= e^{\rho(\log(a))} \int_N f(an)\, dn \\ &= |D(a)| \int_N f(nan^{-1})\, dn \\ &= |D(a)| \int_{K\times N} f(knan^{-1}k^{-1})\, dkdn, \end{aligned}$$

where

$$D(a) = e^{\rho(\log(a))} \prod_{\alpha\in\Sigma^+} |1 - e^{-\alpha(H)}|^{m_\alpha} = \prod_{\alpha\in\Sigma^+} \left| e^{\frac{\alpha(\log a)}{2}} - e^{-\frac{\alpha(\log a)}{2}} \right|^{m_\alpha}.$$

Since the Weyl group permutes the roots, $D(a)$ is invariant under s. So it suffices to show the integral on the right hand side is also invariant. To this end we wish to change variables $(k, n) \mapsto \phi_u(k, n)$, where $u = k_s \in K$ is as in (4.18). So we must check that the map

$$\begin{aligned} g : K \times N &\to \mathbb{R} \\ (k, n) &\mapsto f(knan^{-1}k^{-1}) \end{aligned}$$

has compact support. We have $(k, n) \in \operatorname{Supp}(g)$ if and only if $a^{-1}nan^{-1} = \xi_a(n) \in ak^{-1} \operatorname{Supp}(f)k$. Since ξ_a is a diffeomorphism and since $ak^{-1} \operatorname{Supp}(f)k$ is compact, for each $k \in K$ the set $\{n \in K : (k, n) \in \operatorname{Supp}(g)\}$ is compact. Since K is compact it follows that g has compact support. Thus changing variables $(k_1, n_1) = \phi_u(k, n)$ with $u = k_s$ as above, by Lemma 5.13 we have

$$\begin{aligned} \int_{K\times N} f(kna^s n^{-1}k^{-1})\, dkdn, &= \int_{K\times N} f(kn(uau^{-1})n^{-1}k^{-1})\, dkdn, \\ &= \int_{K\times N} f(uk_1n_1an_1^{-1}k_1^{-1}u^{-1})\, dkdn, \end{aligned}$$

and now changing variables $uk_1 \mapsto k_1$ we obtain

$$\int_{K\times N} f(kna^s n^{-1}k^{-1})\, dkdn = \int_{K\times N} f(k_1n_1an_1^{-1}k_1^{-1})\, dkdn,$$

as desired. Now since both sides are continuous, the functional equation still holds when H is not regular. □

6. Statements of results

6.1. **Notations.** We fix the notations for the remainder of the course. We let G be connected semisimple Lie group with finite centre and $\mathfrak{g}$ be its Lie algebra. By Theorem 4.4 there exists a Cartan involution θ for $\mathfrak{g}$. Accordingly we have the Cartan decomposition (4.5)

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k},$$

where $\mathfrak{p}$ is the -1-eigenspace of θ and $\mathfrak{k}$ is the 1-eigenspace of θ. We recall for future use that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and that the vector space $\mathfrak{p}$ is stable under $\mathrm{Ad}_G(K)$. We let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra, from which we defined in (4.6) the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

where Σ is the set of roots. We fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and we accordingly define the set of positive roots $\Sigma^+ \subset \Sigma$. Then the direct sum

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

is a Lie subalgebra of $\mathfrak{g}$ and we have the the Lie algebra-level Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

By Theorem 3.22 there are connected Lie subgroups N, A, K of G with respective Lie algebras $\mathfrak{n}, \mathfrak{a}, \mathfrak{k}$. As discussed in Proposition 4.20 and § 4.3.1, the group K is compact and A is abelian. Moreover by Theorem 4.24 we have the corresponding Lie group-level Iwasawa decomposition

$$G = NAK.$$

For $g \in G$ we let $A(g) \in \mathfrak{a}$ be the unique element of $\mathfrak{a}$ such that

$$g \in N \exp(A(g)) K.$$

When $g = a \in A$ we might alternatively use the notation $\log a$ for $A(a)$. In Definition 4.31 we defined the Weyl group W as the group generated by the reflections s_α $(\alpha \in \Sigma)$ and we saw that it is finite and permutes the roots. Given a root α its multiplicity is by definition $m_\alpha = \dim \mathfrak{g}_\alpha$. We denote by ρ the half-sum of positive roots counted with multiplicity

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \in \mathfrak{a}^*.$$

For future use, we let $\mathfrak{q} \subset \mathfrak{p}$ be the orthogonal complement of $\mathfrak{a}$ with respect to the Killing form.

In the sequel it shall be more convenient to work with complex-valued functions so we change our previous notation: from now on from now on $C^\infty(G)$ shall denote $C^\infty(G, \mathbb{C})$ and similarly for $C_c(G)$. We let $D_\mathbb{R}(G)$ be the algebra of left-invariant differential operators on G (that we denoted by $D(G)$ in § 3.2), and we accordingly use the new notation

$$D(G) = D_\mathbb{R}(G) \otimes_\mathbb{R} \mathbb{C}.$$

The $\mathbb{R}$-linear action of $D_\mathbb{R}(G)$ on $C^\infty(G)$ extends to a $\mathbb{C}$-linear action of $D(G)$. Similarly we let

$$\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$$

and we extend the Lie bracket on $\mathfrak{g}$ to a $\mathbb{C}$-bilinear map on $\mathfrak{g}_\mathbb{C}$. More generally if $\mathfrak{v}$ is any vector subspace of $\mathfrak{g}$ we let $\mathfrak{v}_\mathbb{C} = \mathfrak{v} \otimes_\mathbb{R} \mathbb{C}$, if $\mathfrak{v}$ is a Lie subalgebra of $\mathfrak{g}$ then $\mathfrak{v}_\mathbb{C}$ is still stable by the complexification of the Lie bracket. It is straightforward to verify that Corollaries 3.19 and 3.20 still hold upon replacing the ground field $\mathbb{R}$ with $\mathbb{C}$. For instance if $X_1, \cdots, X_n$ is a basis of the real vector space $\mathfrak{g}$ then (the image in $\mathfrak{g}_\mathbb{C}$ of) $X_1, \cdots, X_n$ is a basis of the complex vector space $\mathfrak{g}_\mathbb{C}$, and moreover we have the isomorphisms of complex vector spaces

$$D(G) \cong S(\mathfrak{g}_\mathbb{C}) \cong \mathbb{C}[X_1, \cdots, X_n].$$

We recall that for any $g \in G$ we have an operator R_g defined for every function f on G and by

$$R_g f : x \mapsto f(xg).$$

It is easily verified that if $X \in \mathfrak{g}$ and $f \in C^\infty(G)$ then for all $g \in G$ we have

$$(\mathrm{Ad}_G(g)X)f = R_g \circ X \circ R_g^{-1} f,$$

thus $\mathrm{Ad}_G(g)$ extends to a automorphism of the algebra $D(G)$. Concretely

$$\mathrm{Ad}_G(g)D = R_g \circ D \circ R_g^{-1}$$

for all $D \in D(G)$. We let

$$D_K(G) = \{D \in D(G) : \mathrm{Ad}_G(k)D = D \text{ for all } k \in K\},$$

that is, $D_K(G)$ consists of those differential operators $D \in D(G)$ that commute with R_k for all $k \in K$.

6.2. **Spherical functions.** Let $\phi \in C^\infty(G)$ such that $\phi(e) = 1$. Then ϕ is a **spherical function** if

(1) $\phi(k_1 g k_2) = \phi(g)$ for all $g \in G$ and for all $k_1, k_2 \in K$,
(2) for each $D \in D_K(G)$ there exists $\lambda_D \in \mathbb{C}$ such that $D\phi = \lambda_D \phi$.

The theme of this course is that spherical functions on G play the same role in harmonic analysis of G as the complex exponential plays in Fourier theory. However, while the complex exponential is completely understood, at this stage we don't know anything about spherical functions: Do they exist? Can we somehow parametrise them? What analytical properties do they have? As overviewed in the introduction, spherical functions essentially are the spherical matrix coefficients of admissible representations of G. So the classification of spherical functions has deep ties with the classification of spherical admissible representations of G (that was achieved by Langlands). There is some subtlety here related to the reducibility of certain induced representations, but we won't go into it.

For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ define

$$\phi_\nu(g) = \int_K e^{(\rho+i\nu)(A(kg))}\, dk. \tag{6.1}$$

Here the Haar measure on K is normalised such that $\int_K dk = 1$. The integrand is continuous and K is compact so the integral (6.1) converges.

Remark 6.1. *As we shall see below, the function $F_\nu : g \mapsto e^{\nu(A(g))}$ is already a right-K-invariant eigenfunction of each $D \in D_K(g)$. So definition (6.1) is natural: to produce a left-K-invariant eigenfunction, we average F_ν over K. That we are using $\rho + i\nu$ instead of ν is a purely a normalisation question.*

Proposition 6.2. *For all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ the function ϕ_ν is a spherical function.*

Proposition 6.3. *For all $s \in W$ and for all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ we have $\phi_{s\nu} = \phi_\nu$.*

Theorem 6.4 (Harish-Chandra)**.** *Every spherical function is of the form ϕ_ν for some $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. Moreover we have $\phi_\lambda = \phi_\nu$ if and only if $\lambda = s\nu$ for some $s \in W$.*

We can already prove Proposition 6.3

Proof of Proposition 6.3. To prove $\phi_\nu = \phi_{s\nu}$ it suffices to prove that for all $f \in C_c(G)$ we have

$$\int_G \phi_{s\nu}(g) f(g)\, dg = \int_G \phi_\nu(g) f(g) dg.$$

Since moreover $\phi_\nu = \phi_{s\nu}$ are bi-K-invariant, it suffices to prove it for f bi-K-invariant, which we shall henceforth assume. By the Iwasawa integration formula (Theorem 5.8) we have

$$\begin{aligned}\int_G \phi_{s\nu}(g)f(g)\,dg &= \int_K\int_G e^{(\rho+is\nu)(A(kg))}f(g)\,dkdg\\ &= \int_G e^{(\rho+is\nu)(A(g))}f(g)\,dg\\ &= \int_A\int_N e^{(is\nu-\rho)(\log(a))}f(na)da\\ &= \int_A e^{is\nu(\log(a))}F_f(a)da = \int_A e^{i\nu(\log(a))}F_f(a^s)da.\end{aligned}$$

where $F_f(a) = e^{\rho(\log(a))}\int_N f(an)dn$. Now by Theorem 5.15 we have $F_f(a) = F_f(a^s)$, which finishes the proof. □

To prove Proposition 6.1 and Theorem 6.4 we must return to studying the algebra of differential operators, which will be the object of Section 7.

6.3. **The inversion theorem.** We shall denote by $C_c^\infty(G/\!\!/K)$ the subspace of $C^\infty(G)$ consisting of functions f with compact support and satisfying $f(k_1gk_2) = f(g)$ for all $k_1, k_2 \in K$ and for all $g \in G$. If $f \in C_c^\infty(G/\!\!/K)$, its **spherical transform** is by definition the function on $\mathfrak{a}_{\mathbb{C}}^*$ defined by

$$\tilde{f}(\nu) = \int_G f(x)\phi_{-\nu}(x)\,dx.$$

It is a non-abelian generalisation of the Fourier transform, where we integrate a given function against an eigenfunction of the invariant differential operators. The aim of the following results is to describe the image of the spherical transform and to give an inversion formula.

Theorem 6.5 (Spherical inversion theorem)**.** *There exists a holomorphic function c defined on a W-invariant open subset of $\mathfrak{a}_{\mathbb{C}}^*$ containing $\mathfrak{a}^*$ such that c is W-invariant and such that for all $f \in C_c^\infty(G/\!\!/K)$ we have*

$$f(g) = \int_{\mathfrak{a}^*}\tilde{f}(\nu)\phi_\nu(g)\frac{d\nu}{|c(\nu)|^2}$$

for all $g \in G$. Moreover

$$\int_G |f(g)|^2\,dg = \int_{\mathfrak{a}^*}|\tilde{f}(\nu)|^2\frac{d\nu}{|c(\nu)|^2}.$$

An entire function h on $\mathfrak{a}_{\mathbb{C}}^*$ is a **Paley-Wiener function** if there exists $R > 0$ such that for all non-negative integer N there exists a constant C_N such that

$$|h(\nu)| \leq C_N(1 + |\nu|)^{-N} e^{R|\Im(\nu)|}$$

for all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. Here $|\nu|$ denotes any norm on $\mathfrak{a}_{\mathbb{C}}^*$ (a different choice of norm amounts to change the constants R and C_N). We shall denote by $\mathscr{H}_W(\mathfrak{a}_{\mathbb{C}}^*)$ the space of Paley-Wiener functions h on $\mathfrak{a}_{\mathbb{C}}^*$ which furthermore satisfy the functional equation $h(s\nu) = h(\nu)$ for all $s \in W$.

Theorem 6.6 (Paley-Wiener theorem). *The spherical transform is a bijection from $C_c^\infty(G /\!\!/ K)$ to $\mathscr{H}_W(\mathfrak{a}_{\mathbb{C}}^*)$.*

The crucial ingredient is Harish-Chandra's expansion of the spherical functions. The latter controls the analytical behaviour of ϕ_ν, and Theorems 6.6 and 6.5 may be deduced from it using techniques of complex analysis. Harish-Chandra's expansion of the spherical functions will be the object of Section 8.

7. Spherical functions and the algebra $D(G)$

We start with fixing some notations. In the following if $\mathscr{X} \subset D(G)$ we let $\mathscr{X} D(G)$ (respectively $D(G)\mathscr{X}$) be the right (respectively left) ideal of $D(G)$ generated by $\mathscr{X}$. Next, since $D(A)$ is commutative and generated by $\mathfrak{a}$, each linear map $\nu : \mathfrak{a} \to \mathbb{C}$ extends uniquely to a homomorphism of $D(A)$ to $\mathbb{C}$, that we shall denote $D \mapsto D(\nu)$.

Remark 7.1. *This notation might seem counter-intuitive: after all if $H \in \mathfrak{a}$ then the value of ν evaluated at H is denoted by $\nu(H)$, and not $H(\nu)$. The reason we are using this notation (besides following* [Hel00]*) is that we want to emphasise the symmetric algebra structure, that is, the structure of polynomial functions over $\mathfrak{a}^*$. Note that if G is commutative then so are $\mathfrak{g}$ and $D(G)$. Thus in the case of $G = A$ the isomorphism σ from Corollary 3.20 is not only a vector space isomorphism, but an isomorphism between the algebras $S(\mathfrak{a})$ and $D(A)$. Then given $D \in D(A)$, the polynomial $\nu \mapsto D(\nu)$ is nothing but $\sigma^{-1}(D)$. Lemma 7.3 below offers a new perspective on this isomorphism, namely $D(\nu)$ is also the eigenvalue of $a \mapsto e^{\nu(\log(a))}$.*

7.1. Existence of spherical functions.

The aim of this section we prove Proposition 6.2. We start with preparatory lemmas.

Lemma 7.2. *Let $D \in D(G)$. Then there exists a unique element $D_{\mathfrak{a}} \in D(A)$ such that*

$$D - D_{\mathfrak{a}} \in \mathfrak{n}D(G) + D(G)\mathfrak{k}. \tag{7.1}$$

Moreover if $F \in C^\infty(G)$ satisfies $F(ngk) = F(g)$ for all $n \in N, g \in G, k \in K$ then we have

$$(DF)_{|A} = D_{\mathfrak{a}}(F_{|A}). \tag{7.2}$$

The operator $D_{\mathfrak{a}}$ is called the **radial part of D relative to the Iwasawa decomposition**. This is the central ingredient in the proof of Proposition 6.2

Proof. By the decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ there is a basis $X_1, \cdots, X_n$ of $\mathfrak{g}$ such that $X_1, \cdots, X_{\dim(\mathfrak{n})} \in \mathfrak{n}$, $X_{\dim(\mathfrak{n})+1}, \cdots, X_{\dim(\mathfrak{n})+\dim(\mathfrak{a})} \in \mathfrak{a}$, $X_{\dim(\mathfrak{n})+\dim(\mathfrak{a})+1}, \cdots, X_n \in \mathfrak{k}$. By part (1) of Corollary 3.19 we can express uniquely

$$D = \sum_{e_1, \cdots, e_n \geq 0} a_{e_1, \cdots, e_n} X_1^{e_1} \cdots X_n^{e_n}. \tag{7.3}$$

Letting $D_{\mathfrak{a}}$ be the sum of terms which only involve basis elements of $\mathfrak{a}$, it is clear that (7.1) holds. The uniqueness amounts to showing that

$$D(A) \cap (\mathfrak{n}D(G) + D(G)\mathfrak{k}) = \{0\}.$$

This follows from the observation that in the representation (7.3) of a non-zero element of $\mathfrak{n}D(G) + D(G)\mathfrak{k}$, each term must non-trivially involve either a basis element of $\mathfrak{n}$ or of $\mathfrak{k}$ (because $\mathfrak{n}$ and $\mathfrak{k}$ are Lie algebras). To prove (7.2) it suffices to show

$$(DTF)(a) = (XDF)(a) = 0$$

for all $D \in D(G), T \in \mathfrak{k}$ and $X \in \mathfrak{n}$. Since F is right-K-invariant, we immediately have $TF = 0$. Next,

$$(XDF)(a) = \lim_{t \to 0} \frac{(DF)(a \exp(tX)) - (DF)(a)}{t}.$$

But $a \exp(tX) a^{-1} \in N$ and thus by left-invariance of D

$$(DF)(a \exp(tX)) = L(a \exp(tX) a^{-1})(DF)(a) = (DF)(a)$$

whence $(XDF)(a) = 0$. □

Lemma 7.3. *For all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ the function $g \mapsto e^{\nu(A(g))}$ is an eigenfunction of each $D \in D_K(G)$ with eigenvalue $D_{\mathfrak{a}}(\nu)$.*

Proof. Let $F(g) = e^{\nu(A(g))}$ and let $D \in D_K(G)$. Let $D_{\mathfrak{a}}$ be the radial part of D as in Lemma 7.2. Then by A-invariance we have for all $a_1, a_2 \in A$

$$L(a_1)D_{\mathfrak{a}}(F_{|A})(a_2) = D_{\mathfrak{a}}(L(a_1)F_{|A})(a_2) = D_{\mathfrak{a}}(F(a_1)F_{|A})(a_2) = F(a_1)D_{\mathfrak{a}}(F_{|A})(a_2).$$

In particular taking $a_2 = e$ we obtain

$$D_{\mathfrak{a}}(F_{|A})(a_1) = \lambda_D F(a_1),$$

where $\lambda_D = D_{\mathfrak{a}}(F_{|A})(e)$. In fact if $D = H \in \mathfrak{a}$ (so $D_{\mathfrak{a}} = H$) then by (3.9) we have

$$D_{\mathfrak{a}}(F_{|A})(e) = \left(\frac{d}{dt}e^{\nu(A(\exp tH))}\right)_{t=0} = \left(\frac{d}{dt}e^{\nu(tH)}\right)_{t=0} = \nu(H)$$

and so we see that for any $D \in D_K(G)$ we have $\lambda_D = D_{\mathfrak{a}}(\nu)$. Thus by Lemma 7.2 we have

$$(DF)(a) = \lambda_D F(a)$$

for all $a \in A$. By left-N-invariance and right-K-invariance, we have

$$(DF)(g) = \lambda_D F(g)$$

for all $g \in G$, which is the desired result. □

Proof of Proposition 6.2. We have three things to check, namely

(1) $\phi_\nu(e) = 1$,
(2) ϕ_ν is bi-K-invariant,
(3) ϕ_ν is an eigenfunction of each differential operator $D \in D_K(G)$.

Item (1) follows from the normalisation of the Haar measure on K since $A(k) = 0$ for $k \in K$. Similarly the right-K-invariance of ϕ_ν follows from the right-K-invariance of $g \mapsto A(g)$. For the left-K-invariance, if $k_0 \in K$ then changing variables $k \mapsto kk_0^{-1}$ in the integral we obtain $\phi_\nu(k_0 g) = \phi_\nu(g)$. Finally for (3) let $D \in D_K(G)$. Let $F(g) = e^{(\rho+i\nu)(A(g))}$. By Lemma 7.3 we have $(DF) = D_{\mathfrak{a}}(\rho + i\nu)F$. Then

$$D(\phi_\nu)(g) = \int_K (DF)(kg)dk = D_{\mathfrak{a}}(\rho + i\nu)\int_K F(kg)dk,$$

as desired. The interchange of D with the integral symbol is justified because K is compact (the details are left as an exercise). □

Remark 7.4. *We have proved a bit more than Proposition 6.2, namely the proof shows that for all $D \in D_K(G)$ the eigenvalue λ_D associated with ϕ_ν is given by $\lambda_D = D_{\mathfrak{a}}(\rho + i\nu)$.*

7.2. **The algebra $D_K(G)$.** The aim of the subsequent sections is to prove Theorem 6.4. In order to motivate what follows, we give an overview of the strategy of the proof. Let ψ be a spherical function. We want to show $\psi = \phi_\nu$ for some $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. By definition, for each $D \in D_K(G)$ there is a complex number $\lambda_\psi(D)$ such that $D\psi = \lambda_\psi(D)\psi$. Recall that the eigenvalue of ϕ_ν for D is given by $D_{\mathfrak{a}}(\rho + i\nu)$. We want to show two facts, namely

(1) there exists $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ such that for all D we have $\lambda_\psi(D) = D_{\mathfrak{a}}(\rho + i\nu)$,

(2) a spherical function is completely determined by its system of eigenvalues $\lambda_\psi(D)$ – and hence we must have $\psi = \phi_\nu$.

Property (2) follows from a regularity theorem for the solutions of certain differential equations, and we shall not spend too much time on it. On the other hand, the map $D \mapsto \lambda_\psi(D)$ is an homomorphism from $D_K(G)$ to $\mathbb{C}$, which furthermore vanishes on $D_K(G) \cap D(G)\mathfrak{k}$ since ψ is right-K-invariant. Thus to establish (1) it suffices to show that every homomorphisms from $D_K(G)$ to $\mathbb{C}$ that vanishes on $D_K(G) \cap D(G)\mathfrak{k}$ is of the form $D \mapsto D_{\mathfrak{a}}(\rho + i\nu)$ for some $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. To do so we need a more in-depth understanding of the algebra $D_K(G)$. In fact, we proceed in two steps

(1.1) We shall see in Theorem 7.10 below that the quotient $D(G)/(D_K(G) \cap D(G)\mathfrak{k})$ we may be identified with the the subalgebra $D_W(A)$ of $D(A)$ consisting of W-invariant operators. In particular we will be reduced to studying commutative algebras.

(1.2) We will show in Lemma 7.20 below that any homomorphism from $D_W(A) \to \mathbb{C}$ is of the desired form.

The proof of (1.1) makes use of the symmetrisation map that we have seen in Corollary 3.20, as well as of the Chevalley restriction theorem. The later is a deep result whose proof involves techniques of commutative algebra as well as Galois theory. The proof of (1.2) also uses commutative algebra, but is not as involved as the the Chevalley restriction theorem.

We recall that if V is a finite dimensional vector space we denote by $S(V)$ the symmetric algebra of V, and we recall the symmetrisation map $\sigma : S(\mathfrak{g}) \to D(G)$ from Corollary 3.20. Given a non-negative integer d we let $S_d(\mathfrak{g})$ be the subspace of $S(\mathfrak{g})$ consisting of polynomials of total degree at most d, and $D^d(G) = \sigma(S_d(\mathfrak{g}))$.

Lemma 7.5. *We have $D(G) = D(G)\mathfrak{k} \oplus \sigma(S(\mathfrak{p}))$.*

Proof. Given $P \in S(\mathfrak{g})$ we prove by induction on $\deg(P)$ that there exists $Q \in S(\mathfrak{p})$ such that $\sigma(P-Q) \in D(G)\mathfrak{k}$. Let $X_1, \cdots, X_{\dim(\mathfrak{p})}$ be a basis of $\mathfrak{p}$ and $X_{\dim(\mathfrak{p})+1}, \cdots, X_n$ be a basis of $\mathfrak{k}$, so $X_1, \cdots, X_n$ is a basis of $\mathfrak{g}$. If $\deg(P) \leq 1$ then we take Q to be the part of P that does not involve the variables $X_{\dim(\mathfrak{p})+1}, \cdots, X_n$. Now we assume the induction hypothesis is true when $\deg(P) < d$ and we prove it for $\deg(P) = d$. Without loss of generality we may assume $P = X_1^{e_1} \cdots X_n^{e_n}$ with $e_1 + \cdots + e_n = d$. If $e_{\dim(\mathfrak{p})+1} + \cdots + e_n = 0$ we can take $Q = P$. Otherwise $\sigma(P)$ is a linear combination of terms of the form $X_{j_1} \cdots X_{j_d}$ with $X_{j_i} \in \mathfrak{k}$ for at least one i. We have

$$D_i = X_{j_1} \cdots X_{j_d} - X_{j_1} \cdots X_{j_{i-1}} X_{j_{i+1}} X_{j_d} X_{j_i} \in D^{d-1}(G)$$

and thus taking $D \in D^{d-1}(G)$ to be a suitable linear combination of the terms D_i we obtain
$$\sigma(P) - D \in D(G)\mathfrak{k}.$$
By the induction hypothesis there is $Q \in S(\mathfrak{p})$such that $\sigma(Q) - D \in D(G)\mathfrak{k}$ and hence $\sigma(P - Q) \in D(G)\mathfrak{k}$ as requested. Thus we have shown
$$D(G) = D(G)\mathfrak{k} + \sigma(S(\mathfrak{p})).$$
It remains to show
$$D(G)\mathfrak{k} \cap \sigma(S(\mathfrak{p})) = \{0\}.$$
Let $P \in S(\mathfrak{p})$ such that $P \neq 0$. Then there exists a function $h : \mathbb{R}^{\dim(\mathfrak{p})} \to \mathbb{C}$ such that
$$(P(D_1, \cdots, D_{\dim(\mathfrak{p})})h)(0) \neq 0,$$
where D_i is the derivative with respect to the i-th variable. By the Cartan decomposition Theorem 4.25, there exists $f \in C^\infty(G)$ such that
$$f(\exp(t_1X_1 + \cdots + t_{\dim(\mathfrak{p})}X_{\dim(\mathfrak{p})})k) = h(t_1, \cdots, t_{\dim(\mathfrak{p})})$$
for all sufficiently small $t_1, \cdots, t_{\dim(\mathfrak{p})}$ and all $k \in K$. Then by (3.16) we have
$$\sigma(P)f(e) = (P(D_1, \cdots, D_{\dim(\mathfrak{p})})h)(0) \neq 0,$$
and thus $\sigma(P) \notin D(G)\mathfrak{k}$. □

Corollary 7.6. *Let $I_K(\mathfrak{p}) \subset S(\mathfrak{p})$ be the set of $\mathrm{Ad}_G(K)$-invariants. Then we have*
$$D_K(G) = (D_K(G) \cap D(G)\mathfrak{k}) \oplus \sigma(I_K(\mathfrak{p})).$$

Proof. By definition $D_K(G)$ consists of $\mathrm{Ad}_G(K)$-invariant operators in $D(G)$. Since $\mathfrak{k}$ is stable under $\mathrm{Ad}_G(K)$, so is $D(G)\mathfrak{k}$. Moreover $\mathfrak{p}$ is also stable under $\mathrm{Ad}_G(K)$, hence using $\sigma(\mathrm{Ad}_G(g)P) = \mathrm{Ad}_G(g)\sigma(P)$ we deduce that $\sigma(S(\mathfrak{p}))$ is also stable under $\mathrm{Ad}_G(K)$. The corollary now follows by applying $\mathrm{Ad}_G(K)$ to both side of the direct sum in Lemma 7.5. □

We let $D_W(A)$ be the subalgebra of $D(A)$ consisting of W-invariant operators. Since $D(A)$ is commutative, $D_W(A)$ may be identified with the subalgebra $I_W(\mathfrak{a})$of $S(\mathfrak{a})$ consisting of W-invariant polynomials.

Remark 7.7. *Before proceeding any further we discuss some identifications that we shall make from now on. The Killing form B is non-degenerate, and in particular there is an isomorphism between $\mathfrak{p}$ and $\mathfrak{p}^*$ sending $X \in \mathfrak{p}$ to the linear form $Y \mapsto B(X, Y)$. This extends uniquely to an isomorphism between $S(\mathfrak{p})$ and $S(\mathfrak{p}^*)$, henceforth we shall identify these spaces without further comment. Similarly, we identify $S(\mathfrak{a})$ with $S(\mathfrak{a}^*)$. In particular elements P of $S(\mathfrak{p})$ may, and will, be considered as polynomial functions on $\mathfrak{p}$. It then makes sense to consider the restriction $\overline{P}$ of the function P to the subspace $\mathfrak{a}$. Equivalently,* *under our identification, $\overline{P}$ is the unique element of $S(\mathfrak{a})$ such that $P - \overline{P} \in S(\mathfrak{p})\mathfrak{q}$ (where $\mathfrak{q}$ is the B-orthogonal of $\mathfrak{a}$ in $\mathfrak{p}$).*

Note that if $P \in I_K(\mathfrak{p})$ and if $s \in W$ and $k = k_s \in K$ represents s then we have

$$P = \mathrm{Ad}(k)P \in \mathrm{Ad}(k)\overline{P} + S(\mathfrak{p})(\mathrm{Ad}(k)\mathfrak{q})$$

but since $\mathfrak{q}$ is the orthogonal of $\mathfrak{a}$ in $\mathfrak{p}$ and since $\mathrm{Ad}(k)$ preserves $\mathfrak{a}$ and $\mathfrak{p}$, it also preserves $\mathfrak{q}$, and hence by uniqueness $\mathrm{Ad}(k)\overline{P} = \overline{P}$, that is $\overline{P} \in I_W(\mathfrak{a})$.

Theorem 7.8 (Chevalley's restriction theorem)**.** *Any W-invariant polynomial in $S(\mathfrak{a})$ can be extended to an element of $I_K(\mathfrak{p})$.*

In other terms, Chevalley's restriction theorem says that the restriction map $P \mapsto \overline{P}$ is a surjection from $I_K(\mathfrak{p})$ to $I_W(\mathfrak{a})$. We defer the proof to the next section. However we find it illuminating to give an example.

Example 7.9. *Consider $G = \mathrm{SL}_2(\mathbb{R})$. Then $\mathfrak{g}$ is given by 2×2 matrices whose trace vanishes, and the Cartan involution is $X \mapsto -X^\top$. Thus $\mathfrak{p}$ consists of symmetric matrices $X = \left[\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right]$, and $\mathfrak{a}$ consists of matrices of the form $H = \left[\begin{smallmatrix} a & \\ & -a \end{smallmatrix}\right]$ and $K = \mathrm{SO}_2(\mathbb{R})$. The Weyl group has two elements and the non-trivial one acts on $\mathfrak{a}$ via $\left[\begin{smallmatrix} a & \\ & -a \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -a & \\ & a \end{smallmatrix}\right]$. Let P be a W-invariant polynomial in $S(\mathfrak{a})$. We might see as a polynomial in the variable a, and the W-invariance means $P(a) = P(-a)$. By definition its $\mathrm{Ad}_G(K)$-invariant extension $\tilde{P}$ satisfies*

$$\tilde{P}(\mathrm{Ad}_G(k)(H)) = P(H)$$

for all $H \in \mathfrak{a}$ and for all $k \in K$. So to determine $\tilde{P}(X)$ we must determine $H \in \mathfrak{a}$ such that $X = \mathrm{Ad}_G(k)H = kHk^{-1}$ for some $k \in K$. Now observe

$$2a^2 = \mathrm{tr}(H^2) = \mathrm{tr}(X^2) = 2(x^2 + y^2)$$

and thus

$$H = \pm \begin{bmatrix} \sqrt{x^2+y^2} & \\ & -\sqrt{x^2+y^2} \end{bmatrix}.$$

The choice of sign is irrelevant since P is W-invariant. So we find

$$\tilde{P}(X) = P(\sqrt{x^2 + y^2}).$$

At first glance, the right hand side does not seem to necessarily define a polynomial in x, y (or even a smooth function). However since P is even we can write $P(a) = Q(a^2)$ for some polynomial Q and thus

$$\tilde{P}(X) = Q(x^2 + y^2)$$

is indeed a polynomial.

We denote by e^ρ the map $C^\infty(A) \to C^\infty(A)$ defined by $(e^\rho f)(a) = e^{\rho(\log(a))} f(a)$.

Theorem 7.10. *The map*

$$\gamma : D \mapsto e^{-\rho} D_{\mathfrak{a}} \circ e^{\rho}$$

is a surjective homomorphism from $D_K(G)$ *to* $D_W(A)$ *with kernel* $D_K(G) \cap D(G)\mathfrak{k}$.

Proof. First, we need to see that $e^{-\rho} D_{\mathfrak{a}} \circ e^{\rho}$ is indeed an element of $D(A)$. It suffices to check it when $D_{\mathfrak{a}} = H \in \mathfrak{a}$. In this case using Leibniz' law and Lemma 7.3, we have for all $f \in C^{\infty}(A)$

$$e^{-\rho} H(e^{\rho} f) = (H + \rho(H))f.$$

Note that this also implies

$$\deg(e^{-\rho} D_{\mathfrak{a}} \circ e^{\rho}) = \deg(D_{\mathfrak{a}}), \tag{7.4}$$

a fact that we will use below. Next the map γ is clearly linear, so to prove it is an homomorphism we need to prove $\gamma(D^{(1)} D^{(2)}) = \gamma(D^{(1)})\gamma(D^{(2)})$ for all $D^{(1)}, D^{(2)} \in D_K(A)$. Observe that

$$D^{(1)} D^{(2)} - D^{(1)}_{\mathfrak{a}} D^{(2)}_{\mathfrak{a}} = D^{(1)}_{\mathfrak{a}} (D^{(2)} - D^{(2)}_{\mathfrak{a}}) + (D^{(1)}_{\mathfrak{a}} - D^{(1)}_{\mathfrak{a}}) D^{(2)},$$

and $D^{(2)} - D^{(2)}_{\mathfrak{a}}, D^{(1)} - D^{(1)}_{\mathfrak{a}} \in \mathfrak{n} D(G) + D(G)\mathfrak{k}$. Since $D^{(2)} \in D_K(G)$ we have $D^{(2)} T = T D^{(2)}$ for all $T \in \mathfrak{k}$ and hence

$$(D^{(1)} - D^{(1)}_{\mathfrak{a}}) D^{(2)} \in \mathfrak{n} D(G) + D(G)\mathfrak{k}.$$

On the other hand since $[\mathfrak{n}, \mathfrak{a}] \subset \mathfrak{a}$ we have

$$D^{(1)}_{\mathfrak{a}} (D^{(2)} - D^{(2)}_{\mathfrak{a}}) \in \mathfrak{n} D(G) + D(G)\mathfrak{k},$$

and thus

$$(D^{(1)} D^{(2)})_{\mathfrak{a}} = D^{(1)}_{\mathfrak{a}} D^{(2)}_{\mathfrak{a}},$$

from which follows γ is indeed a homomorphism. To prove the kernel is $D_K(G) \cap D(G)\mathfrak{k}$, we start with the following claims.

Claim 1. For $P \in S(\mathfrak{p})$ let $\overline{P}$ be the unique element of $S(\mathfrak{a})$ such that $P - \overline{P} \in S(\mathfrak{p})\mathfrak{q}$, as in Remark 7.7. Then for each nonconstant $P \in I_K(\mathfrak{p})$ we have

$$\deg(\gamma(\sigma(P)) - \overline{P}) < \deg(P).$$

Proof of the claim. Let d be the degree of P. Writing any element $X \in \mathfrak{q}$ as

$$X = Z - \theta Z = 2Z - (Z + \theta Z)$$

where $Z \in \mathfrak{n}$, we see that

$$P - \overline{P} \in \mathfrak{n} S_{d-1}(\mathfrak{g}) + S_{d-1}(\mathfrak{g})\mathfrak{k}.$$

Since $\mathfrak{a}$ is commutative, we have $\sigma(\overline{P}) = \overline{P}$. Moreover observe that if $Q_1 \in S_{d_1}(\mathfrak{g})$ and $Q_2 \in S_{d_2}(\mathfrak{g})$ then

$$\sigma(Q_1Q_2) \in \sigma(Q_1)\sigma(Q_2) + D^{d_1+d_2-1}(G).$$

Thus

$$\sigma(P) - \overline{P} \in \mathfrak{n}D(G) + D(G)\mathfrak{k} + D^{d-1}(G).$$

Taking the radial part and using (7.4), the claim follows. $\square_{claim}$

Claim 2. For $P \in I_K(\mathfrak{p})$ and $\overline{P}$ as above we have $\deg(P) = \deg(\overline{P})$.

Proof of the claim. First observe that if $P \in S(\mathfrak{p})$ then $\deg(P) = \deg(\mathrm{Ad}_G(k)P)$ for all $k \in K$, and in particular if P is $\mathrm{Ad}_G(K)$-invariant then so are its homogeneous components. Thus we may without loss of generality assume that P is homogeneous. Then $\overline{P}$ is homogeneous and either $\overline{P} = 0$ or $\deg(P) = \deg(\overline{P})$. By Lemma 4.27 we have

$$\mathrm{Ad}_G(K)\mathfrak{a} = \mathfrak{p},$$

hence P is determined by its restriction to $\mathfrak{a}$, so if $\overline{P} = 0$ then $P = 0$. $\square_{claim}$

Now let $D \in D_K(G) \cap D(G)\mathfrak{k}$. Since ϕ_ν is right-K-invariant, we have $D\phi_\nu = 0$, whence $(\rho + i\nu)(D_\mathfrak{a}) = 0$ for all $\nu \in \mathfrak{a}_\mathbb{C}^*$. This implies $D_\mathfrak{a} = 0$ and hence $D \in \ker(\gamma)$, as requested. For the reverse inclusion, let $D \in D_K(G)$ such that $\gamma(D) = 0$. By Corollary 7.6 we have $D = D_1 + \sigma(P)$ for some $D_1 \in D_K(G) \cap D(G)\mathfrak{k}$ and $P \in I_K(\mathfrak{p})$. Then $\gamma(\sigma(P)) = 0$ and thus by the Claim 1 if P is nonconstant then $\deg(\overline{P}) < \deg(P)$, which contradicts Claim 2. Thus $D \in D_K(G) \cap D(G)\mathfrak{k}$.

Finally we need to prove that the image of γ is indeed $D_W(A)$. Let $D_K \in D(G)$. Then by Remark 7.4 we have

$$D\phi_\nu = D_\mathfrak{a}(\rho + i\nu)\phi_\nu = \gamma(D)(i\nu)\phi_\nu.$$

Now Proposition 6.3 implies that $\gamma(D)$, viewed as a polynomial on $\mathfrak{a}_\mathbb{C}^*$, is W-invariant, that is $\gamma(D) \in I_W(\mathfrak{a}) = D_W(A)$. For the surjectivity, let $Q \in I_W(\mathfrak{a})$ be a homogeneous polynomial of degree d. We prove by induction on d that there exists $D \in D_K(G)$ such that $\gamma(D) = Q$. For $d = 0$ this is obvious. Let $d > 0$ and by Chevalley's restriction Theorem 7.8 consider $P \in I_K(\mathfrak{p})$ such that $\overline{P} = Q$. By Claim 2 we have $\deg(P) = d$ and thus by Claim 1 we have

$$\deg(\gamma(\sigma(P)) - Q) < d.$$

Since we have just proved $\gamma(\sigma(P)) \in I_W(\mathfrak{a})$ and since $Q \in I_W(\mathfrak{a})$ by assumption, we have

$$\gamma(\sigma(P)) - Q \in I_W(\mathfrak{a})$$

and thus by induction hypothesis there exists $D_1 \in D_K(G)$ such that $\gamma(D_1) = \gamma(\sigma(P)) - Q$. By Corollary 7.6 we have $\sigma(P) \in D_K(G)$, thus taking $D = \sigma(P) - D_1$ finishes the proof. □

7.3. **A proof of Chevalley's restriction theorem.** We give an algebraic proof of Theorem 7.8, following [Hel62, Chap. X, Theorem 6.10]. An analytic proof can be found in [Hel00, Chap. II, Theorem 5.8]. The latter works more generally for smooth functions, and the case of polynomials is derived from the fact that a smooth homogeneous function is a polynomial.

We start with some algebra prerequisites. Let A be an integral domain and $R \subset A$ be a subring. An element $x \in A$ is **integral over** R if there exists a monic polynomial $P \in R[X]$ such that $P(x) = 0$. If every $x \in A$ is integral over R, we say A is **integral over** R. Given an integral domain R, we denote by $\mathrm{Frac}(R)$ its field of fractions. Then R is **integrally closed** if any element of $\mathrm{Frac}(R)$ that is integral over R belongs to R.

Lemma 7.11. *Let A be an integral domain and $R \subset A$ be a subring. Assume $x, y \in A$ are integral over R. Then the ring $R[x, y]$ is integral over R.*

Proof. Since x is integral over R we have $x^n = a_{n-1}x^{n-1} + \cdots + a_0$ for some integer n and $a_0, \cdots, a_{n-1} \in R$, and similarly $y^m = b_{m-1}x^{m-1} + \cdots + b_0$. Therefore $R[x, y]$ is generated by R and the elements $Y_{\mathbf{i}} := x^{i_1}y^{i_2}$ with $0 \le i_1 \le n-1$ and $0 \le i_2 \le m-1$. Therefore given $p \in R[x, y]$ we can write

$$pY_{\mathbf{i}} = \sum_{\mathbf{j}} a_{\mathbf{i},\mathbf{j}} Y_{\mathbf{j}}$$

for some $a_{\mathbf{i},\mathbf{j}} \in R$, that is $AY = 0$, where $A_{\mathbf{i},\mathbf{j}} = \delta_{ii,jj}p - a_{\mathbf{i},\mathbf{j}}$. Multiplying by the transpose comatrix of A we deduce $(\det A)Y = 0$ and hence $\det A = 0$. But $\det A$ is a monic polynomial in p with coefficients in R, so p is integral over R, as desired. □

Lemma 7.12. *Any unique factorisation domain R is integrally closed.*

Proof. Let $x \in \mathrm{Frac}(R)$ and let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in R(X)$ such that $P(x) = 0$. Write $x = \frac{\alpha}{\beta}$ with $\alpha, \beta \in R$ relatively prime. Then we have

$$-\alpha^n = a_{n-1}\alpha^{n-1}\beta + \cdots + a_1\alpha\beta^{n-1} + \beta^n.$$

If β has a prime factor γ then γ must divide α^n thus γ divides α, contradicting the fact that α and β are relatively prime. Thus β is a unit, hence $x \in R$ as desired. $\square$

In particular since the polynomial algebra in n indeterminates $\mathbb{C}[X_1, \cdots, X_n]$ is a unique factorisation domain, the symmetric algebra $S(\mathfrak{p})$ is integrally closed.

Lemma 7.13. *If $P_1, P_2 \in I_K(\mathfrak{p})$ and $P_1 = P_2 Q$ with $Q \in S(\mathfrak{p})$ then $Q \in I_K(\mathfrak{p})$.*

Proof. By $\mathrm{Ad}_G(K)$-invariance we have

$$P_1 = \int_K \mathrm{Ad}_G(k) P_1 \, dk = \int_K \mathrm{Ad}_G(k)(P_2 Q) \, dk = P_2 \int_K \mathrm{Ad}_G(k) Q \, dk$$

and hence

$$Q = \int_K \mathrm{Ad}_G(k) Q \, dk.$$

But the right hand side is manifestly $\mathrm{Ad}_G(K)$-invariant, thus so is Q. $\square$

Lemma 7.14. *The ring $I_K(\mathfrak{p})$ is integrally closed.*

Proof. Let $q \in \mathrm{Frac}(I_K(\mathfrak{p})) \subset \mathrm{Frac}(S(\mathfrak{p}))$ be integral over $I_K(\mathfrak{p})$. Since $I_K(\mathfrak{p})$ is a subring of $S(\mathfrak{p})$, q is in particular integral over $S(\mathfrak{p})$ which as we just saw is integrally closed, thus $q \in S(\mathfrak{p})$. Now write $q = \frac{p_1}{p_2}$ with $p_1, p_2 \in I_K(\mathfrak{p})$. By Lemma 7.13 we have $q \in I_K(\mathfrak{p})$. $\square$

We now let $J \subset I_W(\mathfrak{a})$ be the image of $I_K(\mathfrak{p})$ under the restriction map $P \mapsto \overline{P}$. Proving Chevalley's restriction theorem amounts to prove that $J = I_W(\mathfrak{a})$. By the argument of Claim 2 in the proof of Theorem 7.10, the map $P \mapsto \overline{P}$ is injective on $I_K(\mathfrak{p})$, thus J is isomorphic to $I_K(\mathfrak{p})$, and in particular by Lemma 7.14 J is integrally closed.

Lemma 7.15. *The ring $S(\mathfrak{a})$ is integral over J.*

For $X \in \mathfrak{p}$ let T_X be the restriction of $(\mathrm{ad}\, X)^2$ to $\mathfrak{p}$. Note that T_X is an endomorphism as for all $X, Y \in \mathfrak{p}$ we have

$$\theta([X, [X, Y]] = [\theta X, [\theta X, \theta Y]] = -[X, [X, Y]].$$

In particular we may consider its characteristic polynomial

$$\chi(X; \lambda) = \det(\lambda I - T_X).$$

Note that for all $k \in K$ we have

$$T_{\mathrm{Ad}_G(k) X} = \mathrm{Ad}_G(k) \circ T_X \circ \mathrm{Ad}_G(k)^{-1},$$

and in particular $\chi(X;\lambda)$ is $\mathrm{Ad}_G(K)$-invariant in the variable X.

Lemma 7.16. *Fix $H \in \mathfrak{a}$. The roots of the polynomial $\chi(H;\lambda^2)$ are 0 and the numbers $\alpha(H)$, where $\alpha \in \Sigma$.*

Proof. For $X \in \mathfrak{a}_{\mathbb{C}}$ we have $(\operatorname{ad} H)(X) = [H, X] = 0$ and hence 0 is in particular an eigenvalue of $(\operatorname{ad} H)^2$ with multiplicity $\dim \mathfrak{a}$. Hence 0 is a root of $\chi(H;\lambda^2)$ with multiplicity (at least) $2\dim\mathfrak{a}$. Next let α be a positive root and let $X_\alpha \in \mathfrak{g}$ be a corresponding root vector. Then we have $X_\alpha - \theta X_\alpha \in \mathfrak{p}$ and

$$(\operatorname{ad} H)(X_\alpha) = \alpha(H)X_\alpha,$$
$$(\operatorname{ad} H)(\theta X_\alpha) = -\alpha(H)\theta X_\alpha,$$

thus $(\operatorname{ad} H)^2(X_\alpha - \theta X_\alpha) = \alpha(H)^2(X_\alpha - \theta X_\alpha)$. Using Remark 4.10 we deduce that $\alpha(H)^2$ is an eigenvalue of T_H with multiplicity $m_\alpha = \dim \mathfrak{g}_\alpha$, and hence a root of $\chi(H;\lambda)$ with multiplicity (at least) m_α. Thus $\alpha(H)$ and $-\alpha(H)$ are both roots of $\chi(H;\lambda^2)$ with multiplicity (at least) m_α each. But using Remark 4.10 again we have

$$\begin{aligned}\deg \chi(H;\lambda^2) &= 2\dim\mathfrak{p} = 2\dim\mathfrak{a} + 2\dim\mathfrak{q} = 2\dim\mathfrak{a} + 2\dim\mathfrak{n}\\ &= 2\dim\mathfrak{a} + \sum_{\alpha\in\Sigma} m_\alpha,\end{aligned}$$

and so we have found all the roots of $\chi(H;\lambda^2)$. □

Proof of Lemma 7.15. Since the characteristic polynomial

$$\chi(X;\lambda) = \det(\lambda I - T_X) = \lambda^r + P_{r-1}(X)\lambda^{r-1} + \cdots + P_0(X)$$

of T_X is $\mathrm{Ad}_G(K)$-invariant, the coefficients $P_i(X)$ are themselves $\mathrm{Ad}_G(K)$-invariant polynomial functions on $\mathfrak{p}$, that is (under the identification from Remark 7.7) $P_i \in I_K(\mathfrak{p})$ for all i. In particular their restriction to $\mathfrak{a}$ belongs to J, that is

$$\chi_{|\mathfrak{a}} \in J[\lambda]. \tag{7.5}$$

Now let α be a positive root. By Lemma 7.16 we get the following identity of functions on $\mathfrak{a}$

$$\alpha^{2r} + P_{r-1}\alpha^{2(r-1)} + \cdots + P_0 = 0,$$

which proves that $\alpha \in S(\mathfrak{a})$ is integral over J. Since by Proposition 4.12 every linear function on $\mathfrak{a}$ is a linear combination of positive roots, we have

$$S(\mathfrak{a}) = J[\alpha_{|\mathfrak{a}}]_{\alpha\in\Sigma^+}. \tag{7.6}$$

Using Lemma 7.11 the result follows. □

Lemma 7.17. *Let $H_0 \in \mathfrak{a}$ and $H_1 \in \mathfrak{a}_{\mathbb{C}}$ such that $P(H_0) = P(H_1)$ for all $P \in J$. Then $H_1 = sH_0$ for some $s \in W$.*

Proof. By Lemma 7.16 for any $H \in \mathfrak{a}_{\mathbb{C}}$ the roots of the polynomial $\chi(H;\lambda^2)$ are 0 and the numbers $\pm\alpha(H)$, where α is a positive root. On the other hand from our assumption we have

$$\chi(H_0;\lambda^2) = \chi(H_1;\lambda^2)$$

for all $\lambda \in \mathbb{C}$. Thus $\alpha(H_1) = \alpha(H_0)$ is real for each positive root α, which implies $H_1 \in \mathfrak{a}$. Next we prove there is $k \in K$ such that $H_1 = \mathrm{Ad}_G(k)H_0$. Using Lemma 4.35 this will finish the proof. Assume for contradiction this is not the case. Since the orbits $\mathrm{Ad}_G(K)H_0$ and $\mathrm{Ad}_G(K)H_1$ are both compact and disjoint from each other, there exists a real-valued continuous function on $\mathfrak{p}$ that is identically zero on the orbit $\mathrm{Ad}_G(K)H_0$ and identically one on the orbit $\mathrm{Ad}_G(K)H_1$. By the Stone-Weierstrass approximation theorem, there exists $P \in S(\mathfrak{p})$ such that

$$|P(H)| < \tfrac{1}{3} \text{ for } H \in \mathrm{Ad}_G(K)H_0,$$
$$|P(H)| > \tfrac{2}{3} \text{ for } H \in \mathrm{Ad}_G(K)H_1.$$

Then the polynomial $Q = \int_K \mathrm{Ad}_G(k)P\,dk$ belongs to $I_K(\mathfrak{p})$ but takes different values on H_0 and H_1, contradicting our assumption. □

We now have all the tools in hand to prove Chevalley's restriction theorem.

Proof of Theorem 7.8. Let $K = \mathrm{Frac}(J)$ and $L = \mathrm{Frac}(S(\mathfrak{a}))$. As previously discussed, the characteristic polynomial (7.5) it is completely reducible over L and its roots are 0 and $\pm\alpha_{|\mathfrak{a}}$. By (7.6) we have $L = K[\alpha_{|\mathfrak{a}}]_{\alpha\in\Sigma^+}$. Thus L is the splitting field of $\chi_{|\mathfrak{a}}$ over K, in particular the extension L/K is Galois. Now let $\sigma \in \mathrm{Gal}(L/K)$. Then σ permutes the roots of $\chi_{|\mathfrak{a}}(\lambda)$ and in particular we have $\sigma(S(\mathfrak{a})) = S(\mathfrak{a})$. Now fix $H_0 \in \mathfrak{a}$ and consider the homomorphism

$$\begin{aligned} \epsilon : S(\mathfrak{a}) &\to \mathbb{C} \\ P &\mapsto \sigma(P)(H_0). \end{aligned}$$

In particular its restriction to $\mathfrak{a}$ is a linear form and thus by duality there exists $H_1 \in \mathfrak{a}$ such that

$$\epsilon(H) = B(H, H_1)$$

for all $H \in \mathfrak{a}$. Then we have $\epsilon(P) = P(H_1)$ for all $P \in S(\mathfrak{a})$. In particular for $P \in J$ we have

$$P(H_1) = \epsilon(P) = \sigma(P)(H_0) = P(H_0)$$

and hence by Lemma 7.17 we have $H_1 = sH_0$ for some $s \in W$. Now let $Q \in I_W(\mathfrak{a})$. Then we have

$$\sigma(Q)(H_0) = \epsilon(Q) = Q(H_1) = Q(sH_0) = Q(H_0).$$

Since $H_0 \in \mathfrak{a}$ was arbitrary we deduce $\sigma(Q) = Q$ for all $\sigma \in \mathrm{Gal}(L/K)$ and for all $Q \in I_W(\mathfrak{a})$. By the fundamental theorem of Galois theory we have $I_W(\mathfrak{a}) \subset K = \mathrm{Frac}(J)$. By Lemma 7.15 $I_W(\mathfrak{a}) \subset S(\mathfrak{a})$ is integral over J, and as mentioned above J is integrally closed. This implies $I_W(\mathfrak{a}) = J$ and the proof is finished. □

Remark 7.18. *As a by-product of the proof we have obtained that* $\mathrm{Gal}(L/K) = W$. *Indeed, the inclusion* $W \subset \mathrm{Gal}(L/K)$ *is immediate by definition of* $I_W(\mathfrak{a})$. *Conversely we have shown that for any* $H_0 \in \mathfrak{a}$ *and* $\sigma \in \mathrm{Gal}(L/K)$ *there exists* $s \in W$ *such that*

$$B(\sigma(H), H_0) = B(H, sH_0) = B(sH, H_0)$$

for all $H \in \mathfrak{a}$. *This implies in particular* $\sigma(H) = sH$ *for all* $H \in \mathfrak{a}$, *and since* σ *is completely determined by its restriction to* $\mathfrak{a}$, *the reverse inclusion holds as well.*

7.4. **A proof of Harish-Chandra's theorem.** Theorem 7.10 provides us with a detailed description of the algebra $D_K(G)$. To complete the strategy outlined at the beginning of § 7.2, we need to understand the homomorphisms from $I_W(\mathfrak{a})$ to $\mathbb{C}$. We start with another result from commutative algebra. For our purpose, a **commutative algebra** over $\mathbb{C}$ is a complex vector space A together with a bilinear map $\cdot : A \times A \to A$ such that $(A, \cdot)$ is an integral domain. We recall that if A is a commutative ring, then a **maximal ideal** of A is an proper ideal $\mathscr{I}$ of A not contained in any other proper ideal, and that an ideal of A is maximal if and only if $A/\mathscr{I}$ is a field.

Theorem 7.19. *Let* B *be a finitely generated commutative algebra. Let* $A \subset B$ *be a subalgebra such that* B *is integral over* A. *Then every homomorphism* $\chi : A \to \mathbb{C}$ *extends to a homomorphism* $\tilde{\chi} : B \to \mathbb{C}$.

Proof. If $\chi = 0$ we take $\tilde{\chi} = 0$. Assume $\chi \neq 0$. Then $A/\ker(\chi) \simeq \mathbb{C}$ thus $\ker(\chi)$ is a maximal ideal of A.

Claim 3. There exists a maximal ideal $\mathscr{N}$ of B such that $A \cap \mathscr{N} = \ker(\chi)$.

Proof of the claim. Consider the set S of all ideals $\mathscr{I}$ of B satisfying $\mathscr{I} \cap A \subset \ker(\chi)$. It is partially ordered by inclusion, and every totally ordered subset I has an upper bound $\bigcup_{\mathscr{I} \in I} \mathscr{I}$. By Zorn's lemma, S has a maximal element $\mathscr{N}$. Assume for contradiction $\mathscr{N} \cap A \subsetneq \ker(\chi)$ and let $x \in \ker(\chi) \setminus \mathscr{N}$. Then $\mathscr{N} + Bx$ is an ideal strictly containing $\mathscr{N}$, by maximality of $\mathscr{N}$ in S we must have $(\mathscr{N} + Bx) \cap A \not\subset \ker(\chi)$. So let $y \in (A \cap (\mathscr{N} + Bx)) \setminus \ker(\chi)$. By definition there exists $b \in B$ such that

$$y - bx \in \mathscr{N} \tag{7.7}$$

Since by assumption B is integral over A, there exist $a_0, \cdots, a_{n-1} \in A$ such that

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0.$$

Multiplying by x^n and using (7.7) we deduce

$$y^n + a_{n-1}xy^{n-1} + \cdots + a_0x^n \in \mathscr{N} \cap A \subset \ker(\chi).$$

But since $x \in \ker(\chi)$ we deduce $y^n \in \ker(\chi)$ and thus $y \in \ker(\chi)$, contradicting our assumption. Hence $\mathscr{N} \cap A = \ker(\chi)$. Next we prove $\mathscr{N}$ is a maximal ideal of B. Assume there is a maximal ideal $\mathscr{M}$ of B such that $\mathscr{N} \subsetneq \mathscr{M}$. Then $\mathscr{M} \cap A$ is an ideal of A containing $\ker(\chi)$, and since $\mathscr{N}$ is a maximal element of S we have $\mathscr{M} \notin S$, hence $\mathscr{M} \cap A \neq \ker(\chi)$. But since $\ker(\chi)$ is a maximal ideal of A, it follows $\mathscr{M} \cap A = A$. Now let $b \in B$. Since B is integral over A and since $A \subset \mathscr{M}$, we have $b^n \in \mathscr{M}$ for some positive integer n. Since $\mathscr{M}$ is a maximal ideal we deduce $b \in \mathscr{M}$. Since b was arbitrary we have $B = \mathscr{M}$, a contradiction. Thus $\mathscr{N}$ is a maximal ideal of B. $\square_{claim}$

Now since B is finitely generated let $b_1, \cdots, b_n \in B$ such that $B = \mathbb{C}[b_1, \cdots, b_n]$. Since A contains $\mathbb{C}$ (via the isomorphism $z \mapsto z \cdot 1_A$) we have in particular $B = A[b_1, \cdots, b_n]$. Now since each b_i is integral over A, we have $B = Ax_1 + \cdots + Ax_m$ for some $x_1, \cdots, x_m \in B$ (of the form $x_i = b_1^{d_{1,i}} \cdots b_n^{d_{n,i}}$). Using $A = \mathbb{C} + \ker(\chi)$ and $A \cap \mathscr{N} = \ker(\chi)$ we deduce that the field $B/\mathscr{N}$ is a finite extension of $A/\ker(\chi) \simeq \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, we have $B/\mathscr{N} = \mathbb{C}$. Thus we may extend χ via

$$B \to B/\mathscr{N} = A/\ker(\chi) \xrightarrow{\chi} \mathbb{C},$$

where the first map is the quotient map. $\square$

Lemma 7.20. *The homomorphisms $I_W(\mathfrak{a}) \to \mathbb{C}$ are of the form*

$$\chi_\nu : P \mapsto P(\nu),$$

where $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. Furthermore we have $\chi_\nu = \chi_\lambda$ if and only if $\lambda = s\nu$ for some $s \in W$.

Proof. By Lemma 7.15 and Theorem 7.8, $S(\mathfrak{a})$ is integral over $J = I_W(\mathfrak{a})$. Let $\chi : I_W(\mathfrak{a}) \to \mathbb{C}$ be a homomorphism. By Theorem 7.19, χ extends to a homomorphism $\tilde{\chi} : S(\mathfrak{a}) \to \mathbb{C}$. Now if $X_1, \cdots, X_n$ is a basis of $\mathfrak{a}$ then $\tilde{\chi}$ is completely determined by its value on $X_1, \cdots, X_n$, that is, $\tilde{\chi}$ is given by the evaluation at some $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ and hence $\chi = \chi_\nu$. Now let $\nu, \lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that $P(\nu) = P(\lambda)$ for all $P \in I_W(\mathfrak{a})$. By an argument similar as Lemma 7.17 we have $\lambda = s\nu$ for some $s \in W$, finishing the proof. $\square$

We admit the following.

Fact 1. *Spherical functions are analytic.*

We shall not prove it here, but we give an idea, following [Hel00, page 400]. The Riemannian manifold G/K, has with a G-invariant structure , which is analytic.

Spherical functions on G may be viewed as functions on G/K which are in particular eigenfunctions of the Laplace-Beltrami operator. This operator has analytic coefficients, and is elliptic, and thus by a theorem of Bernstein its eigenfunctions are analytic. Alternatively, we could have included the assumption that ϕ is analytic in our definition of spherical functions, but the argument sketched above shows that this is not necessary.

Proof of Theorem 5.15. We follow the strategy outlined at the beginning of § 7.2. Let ψ be a spherical function. For all $D \in D_K(G)$ let $\lambda_\psi(D)$ be the eigenvalue of D associated to ψ. Then the map $D \mapsto \lambda_\psi(D)$ is a homomorphism of $D_K(G)$ to $\mathbb{C}$. Since ψ is right-K-invariant, this homomorphism vanishes on $D_K(G) \cap D(G)\mathfrak{k}$, and thus gives rise to a homomorphism $\lambda_\psi : D_K(G)/(D_K(G) \cap D(G)\mathfrak{k}) \to \mathbb{C}$. Composing with the inverse of the map γ from Theorem 7.10, we get a homomorphism $\lambda_\psi \circ \gamma^{-1} D_W(A) \to \mathbb{C}$. Since $D_W(A) \simeq I_W(\mathfrak{a})$ by Lemma 7.20 this homomorphism is of the form $D \mapsto D(\rho + i\nu)$ for some $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. Of course for $D \in I_W(\mathfrak{a})$ we have $D = D_{\mathfrak{a}}$ and hence we have shown that $\lambda_\psi(D)$ coincides with the corresponding eigenvalue for ϕ_ν for all $D \in D_K(G)$. Now let $D \in D(G)$. Consider the operator D_0 defined by

$$(D_0 f)(g) = \int_K [(\mathrm{Ad}_G(k)D)f](g)\, dk$$

for all $f \in C^\infty(G)$. Then $D_0 \in D_K(G)$ and moreover since ψ is right-K-invariant and $D\psi$ is left-K-invariant

$$\lambda_\psi(D_0) = (D_0\psi)(e) = \int_K (D\psi)(k)\, dk = (D\psi)(e).$$

In particular using (3.10) and Proposition 3.16, since ψ is analytic it is determined by its eigenvalues and hence equals ϕ_ν. Finally, if $\phi_\nu = \phi_\lambda$ then in particular ϕ_ν and ϕ_λ have the same eigenvalues and thus by Lemma 7.20 $\lambda = s\nu$ for some $s \in W$. □

8. The spherical function expansion

The aim of this section is to state and prove Harish-Chandra's expansion of the spherical function ϕ_ν, which encapsulates some of the analytic properties of ϕ_ν and introduces the important c-function. To this end, we start with introducing a differential operator of central importance, the Laplacian Ω. The Laplacian is left and right G-invariant, and in particular it is an element of $D_K(G)$. Thus ϕ_ν satisfies the differential equation

$$\Omega\phi_\nu = \Omega_{\mathfrak{a}}(\rho + i\nu)\phi_\nu. \tag{8.1}$$

Moreover ϕ_ν is bi-K-invariant, hence by the Cartan decomposition ϕ_ν is completely determined by its restriction $\phi_{\nu|A^+}$ to A^+. From (8.1) we will deduce that $\phi_{\nu|A^+}$ in turn satisfies the differential equation

$$\Delta(\Omega)\Phi_\nu = \Omega_{\mathfrak{a}}(\rho + i\nu)\Phi_\nu, \tag{8.2}$$

where $\Delta(\Omega)$ is the radial part of Ω relative to the Cartan decomposition (that we will introduce in Section 8.1 below). We will make an ansatz on the solutions to (8.2), namely we will look for solutions Φ_ν given by

$$\Phi_\nu(\exp(H)) = e^{(i\nu-\rho)(H)} \sum_{\mu\in\Lambda^+} \Gamma_\mu(\nu)e^{-\mu(H)},$$

where $H \in \mathfrak{a}^+$,

$$\Lambda^+ = \{n_1\alpha_1 + \cdots + n_k\alpha_k : n_1, \cdots , n_k \in \mathbb{Z}_{\geq 0}\},$$

and the coefficients $\Gamma_\mu(\nu)$ are unknown. Here $\alpha_1, \cdots , \alpha_k$ are the positive roots. We will find there exists only one such solution Φ_ν (up to scaling). Furthermore we will explicitly determine the coefficients $\Gamma_\mu(\nu)$. In addition we will show that Φ_ν satisfies not only (8.2), but is also a solution of

$$\Delta(D)\Phi_\nu = D_{\mathfrak{a}}(\rho + i\nu)\Phi_\nu, \tag{8.3}$$

for each $D \in D_K(G)$. Next, for each element s of the Weyl group W the function $\Phi_{s\nu}$ will also satisfy (8.3), and we will show these functions are linearly independent (at least when ν does not belong to certain hyperplanes). Moreover we will show the following.

Proposition 8.1. *Let $\chi : D_K(G) \to \mathbb{C}$ be a homomorphism and let E_χ be the space of functions Φ on A satisfying*

$$\Delta(D)\Phi = \chi(D)\Phi$$

for all $D \in D_K(G)$. Then $\dim(E_\chi) \leq |W|$.

The conclusion of this discussion is that $\phi_{\nu|A}$ is given by a suitable linear combination of the form

$$\sum_{s\in W} c(s\nu)\Phi_{s\nu},$$

where $c(\nu)$ is by definition the Harish-Chandra's c-function. This will provide the desired expansion for ϕ_ν.

Remark 8.2. *In the proof of Proposition 8.1 we will admit that any $\phi \in E_\chi$ is analytic. This follows from a similar argument as the analyticity of the spherical functions, using an elliptic regularity theorem. Alternatively, we prove the weaker statement that the space of* *analytic functions in E_χ has dimension at most $|W|$. This weaker statement is sufficient for our later purpose, since $\phi_{\nu|A}$ is analytic.*

8.1. **The radial part relative to the Cartan decomposition.** In Section 7.1 we have already seen an example of radial part of an operator, namely the radial part for the Iwasawa decomposition. We now introduce the radial part relative to the Cartan decomposition. For a unified definition of "radial parts" from a geometric perspective, see [Hel00, Chap. II § 3]. Given $D \in D_K(G)$, we define an operator $\Delta(\Omega_G)$ on $C^\infty(A^+)$ by

$$\Delta(D)(f_{|A^+}) = (Df)_{|A^+}$$

for all $f \in C^\infty(G /\!\!/ K)$. By Theorem 4.36 if $f_1, f_2 \in C^\infty(G /\!\!/ K)$ coincide on A^+ then $f_1 = f_2$ thus $\Delta(D)$ is well defined. Moreover the restriction to A^+ is a surjection from $C^\infty(G /\!\!/ K)$ to $C^\infty(A^+)$, so $\Delta(D)$ is indeed defined on all of $C^\infty(A^+)$. The operator $\Delta(D)$ is called the **radial part of D relative to the Cartan decomposition**.

8.4. **The Laplacian.** Fix an orthogonal basis $X_1, \cdots, X_{\dim(\mathfrak{g})}$ of $\mathfrak{g}$. We define the **Laplacian on G** by

$$\Omega_G = \sum_{i=1}^{\dim(\mathfrak{g})} \frac{X_i \circ X_i}{B(X_i, X_i)} \in D(G).$$

Remark 8.11. *The Killing form B is not definite on $\mathfrak{g}$, thus there is no orthonormal basis. However we may still diagonalise B, which does ensure the existence of an orthogonal basis $X_1, \cdots, X_{\dim(\mathfrak{g})}$. Moreover B is non-degenerate, and hence if $X_1, \cdots, X_{\dim(\mathfrak{g})}$ is any orthogonal basis then we have $B(X_i, X_i) \neq 0$ for all i, ensuring that our definition makes sense.*

We first check that this definition does not depend on the choice of an orthonormal basis. Observe first that the dual basis is given by

$$X_i^* = \frac{B(X_i, \cdot)}{B(X_i, X_i)}. \tag{8.7}$$

Recall the symmetrisation map σ from Corollary 3.20. Then we have $\Omega_G = \sigma(N)$, where

$$N = \sum_{i=1}^{\dim(\mathfrak{g})} \frac{X_i^2}{B(X_i, X_i)} \in S(\mathfrak{g}).$$

Now let $\nu \in \mathfrak{g}^*$. Recall that since the Killing form is non-degenerate, there exists $A_\nu \in \mathfrak{g}$ (that does not depend on the choice of basis) such that $\nu = B(\cdot, A_\nu)$. By (8.7) we have

$$A_\nu = \sum_{i=1}^{\dim(\mathfrak{g})} \frac{B(X_i, A_\nu)}{B(X_i, X_i)} X_i,$$

and thus

$$N(\nu) = \sum_{i=1}^{\dim(\mathfrak{g})} \frac{\nu(X_i)^2}{B(X_i, X_i)} = \sum_{i=1}^{\dim(\mathfrak{g})} \frac{B(X_i, A_\nu)^2}{B(X_i, X_i)} = B(A_\nu, A_\nu) = \langle \nu, \nu \rangle. \tag{8.8}$$

Hence we have showed N does not depend on the choice of orthogonal basis, and a fortiori $\Omega_G = \sigma(N)$ does not either.

Lemma 8.12. *For all $g \in G$ we have $\mathrm{Ad}_G(g)\Omega_G = \Omega_G$.*

Proof. By definition

$$\mathrm{Ad}_G(g)\Omega_G = \sum_{i=1}^{\dim(\mathfrak{g})} \frac{Y_i \circ Y_i}{B(X_i, X_i)} = \sum_{i=1}^{\dim(\mathfrak{g})} \frac{Y_i \circ Y_i}{B(Y_i, Y_i)},$$

where $Y_i = \mathrm{Ad}_G(g)X_i$ and we have used that the Killing form is Ad_G-invariant. In particular $(Y_i)_i$ is an orthogonal basis and hence by the discussion above the right hand side equals Ω_G. □

Remark 8.13. *Lemma 8.12 states that Ω_G is right-invariant under every element of G. Although we have not talked about universal enveloping algebras, this says that Ω_G is an element of the centre of the universal enveloping algebra.*

The subgroup A is itself a Lie group, and in particular the Laplacian $\Omega_A \in D(A)$ is defined just the same way.

Theorem 8.18. *We have*

$$\Delta(\Omega_G) = \Omega_A + \sum_{\alpha \in \Sigma^+} m_\alpha (\coth \alpha) A_\alpha,$$

where $A_\alpha \in \mathfrak{a}$ is such that $\alpha(H) = B(A_\alpha, H)$ for all $H \in \mathfrak{a}$ and we have identified α with the function $a \mapsto \alpha(\log a)$.

8.5. **The root lattice.** In order to study the Harish-Chandra expansion below (among other things) we need to investigate the root lattice

$$\Lambda = \{n_1\alpha_1 + \cdots n_\ell \alpha_\ell : n_1, \cdots, n_\ell \in \mathbb{Z}\},$$

where $\{\alpha_1, \cdots, \alpha_\ell\} = \Sigma$.

Proposition 8.19. *Let* $\alpha, \beta \in \Sigma$. *Then*

$$2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}.$$

Moreover if $\alpha \neq \beta$ *and* $\alpha - \beta \notin \Sigma$ *then* $\langle \alpha, \beta \rangle \leq 0$.

Proof. It suffices to consider the case when $\alpha \neq \beta$. Moreover without loss of generality we may assume $\alpha - \beta \notin \Sigma$, as otherwise we may replace α with $\alpha - (n-1)\beta$, where n is the first non-negative integer such that $\alpha - n\beta \notin \Sigma$ (such an integer exists considering Σ is finite). Now let $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\beta \in \mathfrak{g}_\beta$ be non-zero root vectors. Recall from the proof of Theorem 4.9 that $H = [X_\beta, \theta X_\beta] \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. Inductively define $e_0 = X_\alpha$ and $e_{n+1} = [X_\beta, e_n] \in \mathfrak{g}_{\alpha+(n+1)\beta}$. We prove by induction that for all integer $n \geq 1$ we have

$$[\theta X_\beta, e_n] = -n\left(\alpha(H) + \frac{n-1}{2}\beta(H)\right) e_{n-1}. \tag{8.10}$$

For $n = 1$ observe that $[\theta X_\beta, X_\alpha] \in \mathfrak{g}_{\alpha-\beta} = \{0\}$ by assumption, and thus

$$[\theta X_\beta, [X_\beta, X_\alpha]] = -[X_\alpha, [\theta X_\beta, X_\beta]] = -\alpha(H)X_\alpha,$$

as desired. Next using the induction hypothesis we have

$$\begin{aligned}
[\theta X_\beta, e_{n+1}] &= [\theta X_\beta, [X_\beta, e_n]] \\
&= -[X_\beta, [e_n, \theta X_\beta]] - [e_n, [\theta X_\beta, X_\beta]] \\
&= -n\left(\alpha(H) + \frac{n-1}{2}\beta(H)\right)[X_\beta, e_{n-1}] - (\alpha(H) + n\beta(H)e_n \\
&= -(n+1)\left(\alpha(H) + \frac{n}{2}\beta(H)\right) e_n.
\end{aligned}$$

Now since Σ is finite, we have $\mathfrak{g}_{\alpha+(n+1)\beta} = \{0\}$ for all n large enough. In particular there is a maximal integer $n \geq 0$ such that $e_n \neq 0$. Thus $[\theta X_\beta, e_{n+1}] = 0$ and using (8.10) we deduce

$$\alpha(H) + \frac{n-1}{2}\beta(H) = 0. \tag{8.11}$$

Now if $A \in \mathfrak{a}$ observe that by (4.4) we have

$$\begin{aligned}
B(H, A) = B([X_\beta, \theta X_\beta], A) &= B([\theta X_\beta, A], X_\beta) \\
&= \beta(A)B(\theta X_\beta, X_\beta),
\end{aligned}$$

from which we deduce $H = B(\theta X_\beta, X_\beta)A_\beta$, and hence

$$\begin{aligned}
\alpha(H) &= B(\theta X_\beta, X_\beta)B(A_\alpha, A_\beta) = B(\theta X_\beta, X_\beta)\langle \alpha, \beta \rangle, \\
\beta(H) &= B(\theta X_\beta, X_\beta)B(A_\beta, A_\beta) = B(\theta X_\beta, X_\beta)\langle \beta, \beta \rangle,
\end{aligned}$$

and reporting this in (8.11) the proposition is proved. □

Corollary 8.20. *Assume $\alpha, c\alpha \in \Sigma$ for some $c \in \mathbb{R}$. Then $c \in \{-2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2\}$.*

Proof. By Proposition 8.19 we have $\frac{2}{c} \in \mathbb{Z}$ and $2c \in \mathbb{Z}$. □

Proposition 8.21. *Let $\alpha_1, \cdots, \alpha_r$ be the simple roots. Then $\alpha_1, \cdots, \alpha_r$ are linearly independent and generate the root lattice Λ, and $r = \dim \mathfrak{a}$.*

Proof. Consider a linear combination of $\alpha_1, \cdots, \alpha_r$ that vanishes. Upon reordering if necessary, we may write it as

$$\sum_{i=1}^{r'} a_i \alpha_i = \sum_{j=r'+1}^{r} a_j \alpha_j,$$

where $a_i \geq 0$ for $0 \leq i \leq r$. Let $\lambda = \sum_{i=1}^{r'} a_i \alpha_i$. Then we have

$$0 \leq \langle \lambda, \lambda \rangle = \sum_{i=1}^{r'} \sum_{j=r'+1}^{r} a_i a_j \langle \alpha_i, \alpha_j \rangle. \tag{8.12}$$

Now since α_i, α_j are simple roots, we have $\alpha_i - \alpha_j \notin \Sigma$, and thus the second statement of Proposition 8.19 implies $\langle \alpha_i, \alpha_j \rangle \leq 0$ for all i, j. Thus the right-hand side of (8.12) is ≤ 0, from which follows $a_i = 0$ for $0 \leq i \leq r$, proving that $\alpha_1, \cdots, \alpha_r$ are linearly independent. It is clear that any root is a $\mathbb{Z}$-linear combination of simple roots, hence the second claim follows. Finally by Proposition 4.12 any element of $\mathfrak{a}^*$ is a $\mathbb{R}$-linear combination of simple roots, hence the set of simple root is a linearly independent generating subset of $\mathfrak{a}^*$, that is a basis, and thus $r = \dim \mathfrak{a}^* = \dim \mathfrak{a}$ as claimed. □

8.6. **Harish-Chandra's expansion of the spherical function.** We now carry out the programme sketched at the beginning of this section. For convenience, we recall the following notations. If $f, g : X \to \mathbb{C}$ are two functions the notation $f(x) \ll g(x)$ means that there exists a constant $c > 0$ such that $|f(x)| \leq c|g(x)|$ for all $x \in X$. Similarly if $h : X \times Y \to \mathbb{C}$ the notation $f(x) \ll_y g(x, y)$ means that for each $y \in Y$ there exists a constant $c_y > 0$ such that $|f(x)| \leq c|g(x, y)|$. The notation $f(x) \asymp g(x)$ means $f(x) \ll g(x) \ll f(x)$.

Proposition 8.22. *Assume $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is such that $2i\langle \lambda, \mu \rangle \neq \langle \mu, \mu \rangle$ for all $\mu \in \Lambda^+ \setminus \{0\}$. Then there exist a unique solution Φ_λ of (8.2) of the form*

$$\Phi_\lambda(\exp(H)) = e^{(i\lambda - \rho)(H)} \sum_{\mu \in \Lambda^+} \Gamma_\mu(\lambda) e^{-\mu(H)}$$

with $\Gamma_0(\lambda) = 1$. Moreover the coefficients Γ_μ are holomorphic on

$$\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : 2i\langle \lambda, \nu\rangle \neq \langle \nu, \nu\rangle \text{ for all } \nu \in \Lambda^+ \setminus \{0\}\}$$

and satisfy

$$|\Gamma_\mu(\lambda)| \ll_H e^{\mu(H)}$$

for all $\mu \in \Lambda^+$ and for all $\lambda \in \mathfrak{a}^* + i\mathfrak{a}_+^*$.

Proposition 8.23. *Assume $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is such that $2i\langle \lambda, \mu\rangle \neq \langle \mu, \mu\rangle$ for all $\mu \in \Lambda^+ \setminus \{0\}$. Then the function Φ_λ from Proposition 8.22 satisfies* (8.3) *for all $D \in D_K(G)$.*

Proposition 8.24. *Assume $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is such that*

$$2i\langle s\lambda, \mu\rangle \neq \langle \mu, \mu\rangle,$$
$$i(s\lambda - \sigma\mu) \notin \Lambda$$

for all $\mu \in \Lambda^+ \setminus \{0\}$ and for all $s \neq \sigma \in W$. Then the function $\Phi_{s\lambda}$ ($s \in W$) are linearly independent.

In accordance with Proposition 8.24, we define

$${}'\mathfrak{a}_{\mathbb{C}}^* = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : 2i\langle s\lambda, \mu\rangle \neq \langle \mu, \mu\rangle \text{ and } i(s\lambda - \sigma\mu) \notin \Lambda \text{ for all } s \neq \sigma \in W, \mu \in \Lambda^+ \setminus \{0\}\}.$$

Theorem 8.25 (Harish-Chandra's expansion)**.** *There exists a function c defined on ${}'\mathfrak{a}_{\mathbb{C}}^*$ such that for all $\lambda \in {}'\mathfrak{a}_{\mathbb{C}}^*$ and for all $H \in \mathfrak{a}^+$ we have*

$$\phi_\lambda(\exp H) = \sum_{s\in W} c(s\lambda) e^{(is\lambda - \rho)(H)} \sum_{\mu \in \Lambda^+} \Gamma_\mu(s\lambda) e^{-\mu(H)},$$

where each Γ_μ is holomorphic on ${}'\mathfrak{a}_{\mathbb{C}}^*$, $\Gamma_0(\lambda) = 1$ and

$$|\Gamma_\mu(\lambda)| \ll_H e^{\mu(H)} \tag{8.13}$$

for all $\mu \in \Lambda^+$ and for all $\lambda \in \mathfrak{a}^* + i\mathfrak{a}_+^*$.

Proof. Let $\lambda \in {}'\mathfrak{a}_{\mathbb{C}}^*$ and define an homomorphism $\chi : D_K(G) \to \mathbb{C}$ by $\chi(D) = D_{\mathfrak{a}}(\rho + i\lambda)$. Then we have $D\phi_\lambda = \chi(D)\phi_\lambda$ for all $D \in D_K(G)$. Thus by definition of $\Delta(D)$, we have $\phi_{\lambda|A^+} \in E_\chi$. Now by Proposition 8.1 we have $\dim(E_\chi) \leq |W|$. On the other hand, Propositions 8.23 and 8.24 provide $|W|$ linearly independent elements of E_χ (and hence a basis), namely the functions $\Phi_{s\lambda}$ ($s \in W$). It follows that $\phi_{\lambda|A^+}$ can be written as a linear combination

$$\phi_{\lambda|A^+} = \sum_{s \in W} c_s(\lambda)\Phi_{s\lambda}.$$

Now using that $\phi_\lambda = \phi_{\sigma\lambda}$ for all $\sigma \in W$ we deduce $c_{e_W}(\sigma\lambda) = c_\sigma(\lambda)$ and, setting $c(\lambda) = c_{e_W}(\lambda)$ the theorem is proved. □

9. The c-function

The c-function encapsulates much of the analytic behaviour of the spherical functions. In particular, its asymptotic properties and the localisation of its zeroes are the main ingredients for the proof of the inversion theorem and the Paley-Wiener theorem. Although Harish-Chandra's original article proceeded in a different way, the most satisfactory account of the properties of the c-function is given by the Gindikin-Karpelevic formula. Let Σ_0 be the subset of Σ consisting of roots that are not a non-trivial integer multiple of another root, and let $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$. For $\alpha \in \Sigma_0$, let $\alpha_0 = \frac{\alpha}{\langle \alpha,\alpha \rangle}$.

Theorem 9.1 (Gindikin-Karpelevic formula)**.** *For all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have*

$$c(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{\Gamma\left(\frac{1+\langle i\lambda,\alpha_0\rangle}{2}\right)\Gamma\left(\frac{\langle i\lambda,\alpha_0\rangle}{2}\right)}{\Gamma\left(\frac{1+\frac{m_\alpha}{2}+\langle i\lambda,\alpha_0\rangle}{2}\right)\Gamma\left(\frac{m_{2\alpha}+\frac{m_\alpha}{2}+\langle i\lambda,\alpha_0\rangle}{2}\right)},$$

where c_0 is such that $c(-i\rho) = 1$.

Remark 9.2. *The Γ function has no zeroes, and has poles at each integer $k \leq 1$. Therefore the Gindikin-Karpelevic formula provides an analytic continuation of $c(\lambda)$ to all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that $\langle \lambda, \alpha_0 \rangle \notin i\mathbb{Z}_{\geq 0}$.*

Remark 9.3. *Using that $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ for all $z \in \mathbb{C}$, the theorem implies in particular that $c(\lambda)c(-\lambda) = |c(\lambda)|^2$ whenever $\lambda \in \mathfrak{a}^* \setminus \{0\}$.*

Remark 9.4. *Since the Weyl group permutes Σ_0, the function $\lambda \mapsto c(\lambda)c(-\lambda)$ is W-invariant.*

We defer the proof to letter sections. As a consequence of the Gindikin-Karpelevic formula, we have the following estimate.

Proposition 9.5. *There are constants C_1 and C_2 such that for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that $\Re(i\lambda) \in \mathfrak{a}_+^*$ we have*

$$|c(\lambda)|^{-1} \leq C_1 + C_2|\lambda|^{\frac{\dim N}{2}}$$

Proof. For any fixed $\epsilon > 0$ and any $s \in \mathbb{C}$ we have the following estimate as $|z| \to \infty$ with $|\arg z| \leq \pi - \epsilon$

$$\frac{\Gamma(z+s)}{\Gamma(z)} \sim z^s$$

(where the right hand side is defined by $z^s = e^{s \log z}$ and log is the principal value of the logarithm). Thus each factor in the Gindikin-Karpelevic formula is

$$\asymp \langle i\lambda, \alpha_0 \rangle^{-\frac{m_\alpha + m_{2\alpha}}{2}}$$

as $|\lambda| \to \infty$ with $\Re(i\lambda) \in \mathfrak{a}_+^*$. Using Corollary 8.20 we have

$$\sum_{\alpha \in \Sigma_0^+} (m_\alpha + m_{2\alpha}) = |\Sigma^+| = \dim \mathfrak{n} = \dim N$$

and hence the result follows. $\square$

9.1. **Example:** $\mathrm{SL}_2(\mathbb{R})$**.** In this section we write down explicitly the integral representation of the spherical functions for $G = \mathrm{SL}_2(\mathbb{R})$, the Laplacian, and the Harish-Chandra expansion.

9.1.1. *The Iwasawa decomposition.* Recall that the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ is given by the set of 2×2 matrices whose trace vanishes. From Examples 4.6 and 4.11 we may take $\mathfrak{k}$, $\mathfrak{a}$ $\mathfrak{n}$ to be the one-dimensional subspaces spanned by $\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, respectively. Taking the usual matrix exponential, the corresponding subgroups are $K = \mathrm{SO}_2 = \{\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right] : \theta \in (-\pi, \pi)\}$, $A = \{\left[\begin{smallmatrix} y & \\ & y^{-1} \end{smallmatrix}\right] : y > 0\}$ and $N = \{\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right] : x \in \mathbb{R}\}$. Then calculating

$$\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right] \begin{bmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{bmatrix} \left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right] = \begin{bmatrix} y\cos\theta - xy^{-\frac{1}{2}}\sin\theta & y^{\frac{1}{2}}\sin\theta + xy^{-\frac{1}{2}}\cos\theta \\ -y^{-\frac{1}{2}}\sin\theta & y^{-\frac{1}{2}}\cos\theta \end{bmatrix} \tag{9.1}$$

we deduce that for $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2$ we have $A(g) = \left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right]$ where

$$H = -\frac{1}{2}\log(c^2 + d^2). \tag{9.2}$$

9.1.2. *The roots.* Recall from Example 4.11 that the roots are the two linear maps $\alpha_{12} : \left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right] \mapsto 2H$ and $\alpha_{21} = -\alpha_{12}$. In this case the root α_{21} is positive and α_{12} is negative. Thus $\rho = \frac{1}{2}\alpha_{12} : \left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right] \mapsto H$.

9.1.3. *The integral representation.* Now let $a = \begin{bmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{bmatrix} \in A$. Then for $k = \left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]$ we deduce from 9.2 that $A(ka) = \left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right]$ where

$$H = -\frac{1}{2}\log(y\sin^2\theta + y^{-1}\cos^2\theta).$$

Now given $s \in \mathbb{C}$ let $\lambda_s \in \mathfrak{a}_\mathbb{C}^*$ given by $\lambda_s : \left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right] \mapsto sH$. Then we have

$$(\rho + i\lambda_s)A(ka) = -\frac{1 + is}{2}\log(y\sin^2\theta + y^{-1}\cos^2\theta),$$

To avoid cumbersome notations, let $\phi_s = \phi_{\lambda_s}$. Then using previous display we have

$$\begin{aligned}\phi_s(a) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(y\sin^2\theta + y^{-1}\cos^2\theta)^{-\frac{1}{2}-\frac{is}{2}}\,d\theta \\ &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(y\sin^2\theta + y^{-1}\cos^2\theta)^{-\frac{1}{2}-\frac{is}{2}}\,d\theta.\end{aligned}$$

This integral may be expressed in terms of the associated Legendre function of the first kind (see [GR15, Equation 3.666. 2]). Changing variables $\theta = \varphi + \frac{\pi}{2}$ and using periodicity, we immediately obtain the relation

$$\phi_s(a) = \phi_s(a^{-1}).$$

On the other hand changing variables $\theta = \arctan(y^2\tan\varphi)$ we have

$$y\sin^2\theta + y^{-1}\cos^2\theta = (y^{-1}\sin^2\varphi + y\cos^2\varphi)^{-1}$$

and

$$d\theta = \frac{d\varphi}{y^{-1}\sin^2\varphi + y\cos^2\varphi},$$

thus we obtain

$$\phi_s(a) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(y^{-1}\sin^2\varphi + y\cos^2\varphi)^{-\frac{1}{2}+\frac{is}{2}}\,d\varphi = \phi_{-s}(a^{-1}) = \phi_{-s}(a),$$

which is Harish-Chandra's functional equation for the non-trivial element of the Weyl group.

9.1.4. *The Laplacian.* Recall from Remark 4.2 that the Killing on $\mathfrak{g}$ form is given by $B(X,Y) = 4\operatorname{tr}(XY)$. Let us pick the orthogonal basis of $\mathfrak{g}$ given by the three vectors

$$Z = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right], \quad Q = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right], \quad H = \left[\begin{smallmatrix} 1 & \\ & -1\end{smallmatrix}\right].$$

We express the corresponding elements of $D(G)$ in terms of the coordinates (9.1).

Lemma 9.6. *We have*

$$\begin{aligned}Z &= \frac{\partial}{\partial\theta}, \\ Q &= 2y\cos(2\theta)\frac{\partial}{\partial x} + 2y\sin(2\theta)\frac{\partial}{\partial y} - \cos(2\theta)\frac{\partial}{\partial\theta}, \\ H &= -2y\sin(2\theta)\frac{\partial}{\partial x} + 2y\cos(2\theta)\frac{\partial}{\partial y} + \sin(2\theta)\frac{\partial}{\partial\theta}.\end{aligned}$$

Proof. Let f be a smooth function on $\mathrm{SL}_2(\mathbb{R})$. We define a smooth function

$$F : \mathbb{R}^3 \to \mathbb{C}$$

given by $F(x, y, \theta) = f(g)$ if $g = \left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]\begin{bmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{bmatrix}\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]$. Observe that

$$\exp(tZ) = \left[\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right].$$

Thus $f(g\exp(tZ)) = F(x, y, \theta + t)$ and the first claim follows. Next let

$$(x_Q(t), y_Q(t), \theta_Q(t))$$

be the Iwasawa coordinates of $g\exp(tQ)$. Then by the chain rule we have

$$Qf = \left(x_Q'(0)\frac{\partial}{\partial x} + y_Q(0)'\frac{\partial}{\partial y} + \theta_Q'(0)\frac{\partial}{\partial \theta} \right) F$$

Observe that

$$\exp(tQ) = \left[\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{smallmatrix}\right],$$

and

$$\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]\left[\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{smallmatrix}\right] = \begin{bmatrix} \cosh t + \sinh t\sin(2\theta) & \sinh t\cos(2\theta) \\ \sinh t\cos(2\theta) & \cosh t - \sinh t\sin(2\theta) \end{bmatrix}\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]. \tag{9.3}$$

Let $(x_s(t), y_s(t), \theta_s(t))$ be the Iwasawa coordinates of

$$\begin{bmatrix} \cosh t + \sinh t\sin(2\theta) & \sinh t\cos(2\theta) \\ \sinh t\cos(2\theta) & \cosh t - \sinh t\sin(2\theta) \end{bmatrix},$$

so that $\theta_Q(t) = \theta + \theta_s(t)$, $y_Q(t) = yy_s(t)$ and $x_Q(t) = x + yx_s(t)$. Thus comparing (9.3) with (9.1) we have

$$\theta_Q(t) = \theta + \arctan\left(-\frac{\sinh t\cos(2\theta)}{\cosh t - \sinh t\sin(2\theta)} \right).$$

In particular $\theta_Q'(0) = -\cos(2\theta)$. Similarly we have

$$y_Q(t) = \frac{y}{\sinh^2 t\cos^2(2\theta) + (\cosh t - \sinh t\sin(2\theta))^2}$$

and hence $y_Q'(t) = 2y\sin(2\theta)$. Finally by differentiating (9.1) and using

$$x_s(0) = 0, \quad y_s(0) = 1, \quad \theta_s(0) = 0, \quad \theta_s'(0) = -\cos(2\theta)$$

we find that

$$-\cos(2\theta) + x_s'(0) = \cos(2\theta),$$

and hence $x_Q'(0) = 2y\cos(2\theta)$. The second claim follows. Similar calculations give the result for H. $\square$

Corollary 9.7. *The Laplacian on G is given by*

$$\Omega_G = \frac{y^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{y}{2}\frac{\partial^2}{\partial x \partial \theta}.$$

Proof. This follows by calculating $\frac{Z\circ Z}{B(Z,Z)} + \frac{Q\circ Q}{B(Q,Q)} + \frac{H\circ H}{B(H,H)}$ explicitly. □

To compute $\Omega_{\mathfrak{a}}$ note that $Z \in \mathfrak{k}$,$H \in \mathfrak{a}$, $Q = X - Z$ where $X = \left[\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{n}$ and

$$\begin{aligned} Q \circ Q &= (X - Z) \circ (X - Z) \\ &= X \circ X + Z \circ Z - Z \circ X - X \circ Z \\ &= X \circ X + Z \circ Z - 2X \circ Z + [X, Z] \\ &= X \circ X + Z \circ Z - 2X \circ Z - 2H \in -2H \in D(G)\mathfrak{k} + \mathfrak{n}D(G) \end{aligned}$$

and hence we have

$$\Omega_{\mathfrak{a}} = \frac{1}{8}H^2 - \frac{1}{4}H$$

and in particular for $\lambda = \lambda_s$ as above,

$$\Omega_{\mathfrak{a}}(\rho + i\lambda_s) = \frac{1}{8}\big((\rho + i\lambda_s)(H)\big)^2 - \frac{1}{4}(\rho + i\lambda_s)(H) = -\frac{1}{8}\left(1 + s^2\right).$$

9.1.5. *The differential equation.* Let $g \in G$. By the Cartan decomposition we may write $g = k_1 a k_2$ with $k_1, k_2 \in K$ and $a = \begin{bmatrix} r^{\frac{1}{2}} & \\ & r^{-\frac{1}{2}} \end{bmatrix} \in A$. Moreover the requirement that $a \in A^+$ is equivalent to $r > 1$ On the other hand we may write g using the Iwasawa decomposition (9.1). Then we have

$$r + r^{-1} = \mathrm{tr}(g^\top g) = \frac{1 + x^2 + y^2}{y}.$$

Thus if f is any bi-K-invariant function on G then we have

$$f(g) = F\left(\frac{1 + x^2 + y^2}{y}\right), \tag{9.4}$$

where

$$F(t) = f\left(\begin{bmatrix} r(t)^{\frac{1}{2}} & \\ & r(t)^{-\frac{1}{2}} \end{bmatrix}\right),$$

and

$$r(t) = \frac{t + \sqrt{t^2 - 4}}{2}.$$

Combining Corollary 9.7 and (9.4) we obtain

$$(\Omega_G f)(a) = (r + r^{-1})F'\left(r + r^{-1}\right) + \frac{1}{2}(r - r^{-1})^2 F''\left(r + r^{-1}\right).$$

In particular if f is in addition an eigenfunction of the Laplacian Ω_G with eigenvalue $-\frac{1}{8}\left(1 + s^2\right)$ say, then we obtain the differential equation

$$\frac{1}{2}(r - r^{-1})^2 F''\left(r + r^{-1}\right) + (r + r^{-1})F'\left(r + r^{-1}\right) + \frac{1}{8}(1 + s^2)F(r + r^{-1}) = 0.$$

Finally changing variable $a = \left[\begin{smallmatrix} e^H & \\ & e^{-H} \end{smallmatrix}\right]$ so $r = e^{2H}$ and setting $G(H) = f(a) = F(e^{2H} + e^{-2H})$ this becomes

$$G''(H) + 2\coth(2H)G'(H) + (1 + s^2)G(H) = 0. \tag{9.5}$$

Compare with Theorem 8.18.

9.1.6. *Harish-Chandra's expansion.* Let us look for a solution $G = G_s$ of (9.5) of the form

$$G_s(H) = e^{(is-1)H} \sum_{n \geq 0} \Gamma_n(s) e^{-2nH}$$

with $\Gamma_0(s) = 1$. Using the series expansion

$$\coth H = 1 + 2\sum_{k \geq 1} e^{-2kH},$$

differentiating (9.5) termwise, and equating the coefficients of e^{-2nH} we obtain the recursion relation

$$n(n - is)\Gamma_n(s) = \sum_{\substack{m+2k=n \\ k \geq 1}} (2m + 1 - is)\Gamma_m(s).$$

This defines the coefficients $\Gamma_n(s)$ uniquely provided that $s \in \mathbb{C} \setminus 2i\mathbb{Z}$. Moreover one checks by induction that for all $H_0 > 0$ and $s \in \mathbb{C} \setminus 2i\mathbb{Z}$ we have

$$|\Gamma_n(s)| \ll_{s,H_0} e^{nH_0} \tag{9.6}$$

and thus, conversely, $G_s(H)$ is well-defined and satisfies (9.5). Now G_{-s} also satisfies (9.5). In addition we have $\Gamma_2(s) = \frac{1-is}{2(2-is)}$ and $\Gamma_2(-s) = \frac{1+is}{2(2+is)} \neq \Gamma_2(s)$ for $s \neq 0$. Thus the two solutions G_s and G_{-s}, which are defined for $s \in \mathbb{C} \setminus 2i\mathbb{Z}$, are linearly independent. By general theory the space of solutions of (9.5) has dimension 2, and since $H \mapsto \phi_s\left(\exp\left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right]\right)$ is also a solution of (9.5) and satisfies $\phi_{-s} = \phi_s$ we have for all $s \in \mathbb{C} \setminus 2i\mathbb{Z}$ and for all $H > 0$

$$\phi_s\left(\exp\left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right]\right) = c(s)G_s(H) + c(-s)G_{-s}(H),$$

where c is the c-function. To determine the c-function explicitly, note that

$$e^{(1-is)H}\phi_s\left(\exp\left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right]\right) = c(s)\sum_{n\geq 0}\Gamma_n(s)e^{-2nH} + c(-s)e^{-2isH}\sum_{n\geq 0}\Gamma_n(s)e^{-2nH}$$

Now take $s \in \mathbb{C}\setminus 2i\mathbb{Z}$ such that $\Re(is) > 0$ and let $H \to \infty$. Using (9.6) with $H_0 = 1$, we may take the limit term by term, and deduce

$$\lim_{H\to\infty} e^{(1-is)H}\phi_s\left(\exp\left[\begin{smallmatrix} H & \\ & -H \end{smallmatrix}\right]\right) = c(s).$$

On the other hand, using the functional equation $\phi_s = \phi_{-s}$ and the integral representation of ϕ_{-s}, by dominated convergence the above limit is

$$\lim_{H\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}(\sin^2\theta + e^{-4H}\cos^2\theta)^{-\frac{1}{2}+\frac{is}{2}}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sin\theta)^{-1+is}\,d\theta.$$

The last integral can be evaluated by [GR15, 3.621 1.], yielding

$$c(s) = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{is}{2}\right)}{\Gamma\left(\frac{1+is}{2}\right)}.$$

Compare with Theorem 9.1.

10. The Paley-Wiener theorem and the inversion theorem

We start preparing the stage for the proofs of Theorems 6.5 and 6.6.

10.1. **Some structural features.** We start with collecting some simple observations on the spherical function and the structural features of the spherical transform.

Lemma 10.1. *For any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and for any $x, y \in G$ we have*

$$\int_K \phi_\lambda(xky)\,dk = \phi_\lambda(x)\phi_\lambda(y).$$

Remark 10.2. *In fact, this property is sometimes taken as the definition of spherical functions.*

Proof. By definition we have

$$\int_K \phi_\lambda(xky)\,dk = \int_K\int_K e^{(\rho+i\lambda)(A(k_1xky))}\,dk_1dk.$$

Now write $k_1x = n\exp(A(k_1x))k_2$ with $n \in N, k_2 \in K$, and observe that $A(k_1xky) = A(k_1x) + A(k_2ky)$. Thus changing variables $k_2k \mapsto k$ we obtain the result. □

Lemma 10.3. *For all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and for all $g \in G$ we have $\phi_{-\lambda}(g^{-1}) = \phi_\lambda(g)$.*

Proof. By definition we have

$$\phi_{-\lambda}(g^{-1}) = \int_K e^{(\rho - i\lambda)(A(kg^{-1}))}\, dk.$$

Now write the Iwasawa decomposition of kg^{-1} as $kg^{-1} = \mathbf{n}\exp(A(kg^{-1}))\mathbf{k}$ where $\mathbf{n} = \mathbf{n}(kg^{-1}) \in N$ and $\mathbf{k} = \mathbf{k}(kg^{-1}) \in K$. Then using

$$\begin{aligned} 0 &= A(k) = A(kg^{-1}g) \\ &= A(\mathbf{n}\exp(A(kg^{-1}))\mathbf{k}g) = A(kg^{-1}) + A(\mathbf{k}g) \end{aligned}$$

we deduce

$$\phi_{-\lambda}(g^{-1}) = \int_K e^{(-\rho + i\lambda)(A(\mathbf{k}(kg^{-1})g))}\, dk.$$

thus to conclude it suffices to show that for any continuous function F on K we have

$$\int_K F(\mathbf{k}(kg^{-1}))\, dk = \int_K e^{2\rho(A(kg))} F(k)\, dk. \tag{10.1}$$

Henceforth let F_1, F_2 be continuous, compactly supported functions on N and A, respectively such that

$$\int_{N\times A} F_1(n)F_2(a)e^{-2\rho\log a}\, dadn \neq 0.$$

Define $f(nak) = F_1(N)F_2(a)F(k)$ for all $n \in N, a \in A, k \in K$. Then by Theorem 5.8 we have

$$\int_G f(x)\, dx = \int_{N\times A\times K} F_1(n)F_2(a)F(k)e^{-2\rho\log a}\, dkdadn. \tag{10.2}$$

On the other hand by changing variables we have

$$\int_G f(x)\, dx = \int_G f(xg^{-1})\, dx$$

Writing

$$nakg^{-1} = [na\mathbf{n}a^{-1}][a\exp(A(kg^{-1}))]\mathbf{k}$$

and changing variable $na\mathbf{n}(kg^{-1})a^{-1} \mapsto n$, $a\exp(A(kg^{-1})) \mapsto a$ we have by Theorem 5.8 again

$$\int_G f(xg^{-1})\, dx = \int_{N\times A\times K} F_1(n)F_2(a)F(\mathbf{k}(kg^{-1}))e^{-2\rho\log a + 2\rho(A(kg^{-1}))}\, dkdadn. \tag{10.3}$$

Comparing (10.2) and (10.3), factoring out $\int_{N\times A} F_1(n)F_2(a)e^{-2\rho\log a}\, dadn$, and replacing $F(k)$ with $e^{2\rho(A(kg))}F(k)$ equation (10.1) follows. □

Corollary 10.4. *Let $g, h \in C_c^\infty(G /\!/ K)$ and let $f = g * h$, where $g * h$ denotes the convolution. Then for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have*

$$\tilde{f}(\lambda) = \tilde{g}(\lambda)\tilde{h}(\lambda),$$

Moreover let $h^*(x) = \overline{h(x^{-1})}$. Then for all $\lambda \in \mathfrak{a}^*$ we have

$$\tilde{h^*}(\lambda) = \overline{\tilde{h}(\lambda)}.$$

Proof. We start with the spherical transform of the convolution. Using Lemma 10.1 and K-invariance we have

$$\begin{aligned}\tilde{f}(\lambda) &= \int_G (g * h)(x)\phi_\lambda(x)\,dx = \int_G \int_G g(y)h(y^{-1}x)\phi_\lambda(x)\,dydx \\ &= \int_G \int_G \int_K g(y)h(x)\phi_\lambda(ykx)\,dkdydx \\ &= \int_G \int_G g(y)h(x)\phi_\lambda(y)\phi_\lambda(x)\,dydx = \tilde{g}(\lambda)\tilde{h}(\lambda).\end{aligned}$$

Next let $\lambda \in \mathfrak{a}^*$. Then using Lemma 10.3 we have

$$\begin{aligned}\tilde{h^*}(\lambda) &= \int_G \overline{h(x^{-1})}\phi_\lambda(x)\,dx \\ &= \int_G \overline{h(x)}\phi_\lambda(x^{-1})\,dx = \int_G \overline{h(x)}\phi_{-\lambda}(x)\,dx,\end{aligned}$$

and the second claim follows upon observing that for $\lambda \in \mathfrak{a}^*$ we have $\phi_{-\lambda} = \overline{\phi_\lambda}$. □

Remark 10.5. *Let $\mathscr{A}$ be the image of $C_c^\infty(G /\!/ K)$ under the map $f \mapsto \tilde{f}_{|\mathfrak{a}^*}$. Then Corollary 10.4 implies that $\mathscr{A}$ is an algebra (under the usual pointwise multiplication of functions) that is closed by complex conjugation.*

10.2. **A localisation property.** An entire function h on $\mathfrak{a}_{\mathbb{C}}^*$ is a Paley-Wiener function **of order** $R > 0$ such that for all non-negative integer M there exists a constant C_M such that

$$|h(\nu)| \leq C_M(1 + |\nu|)^{-M} e^{R|\Im(\nu)|} \tag{10.4}$$

for all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. Here the norm is the one induced by the Killing form. Let us write the Harish-Chandra expansion of the spherical function as

$$\phi_\lambda(\exp H) = \sum_{\mu \in \Lambda^+} \Psi_\mu(\lambda, H) \tag{10.5}$$

for $\lambda \in {}'\mathfrak{a}_{\mathbb{C}}^*$ and $H \in \mathfrak{a}^+$, where

$$\Psi_\mu(\lambda, H) = \sum_{s\in W} c(s\lambda)\Gamma_\mu(s\lambda)e^{(is\lambda-\rho-\mu)(H)}$$

for all $\mu \in \Lambda^+$. The proofs of inversion theorem and of the Paley-Wiener theorem both rest on the following localisation property.

Proposition 10.6. *Let h be a W-invariant Paley-Wiener function of order $R > 0$ on $\mathfrak{a}_{\mathbb{C}}^*$. Than for every $\mu \in \Lambda^+$ and for every $H \in \mathfrak{a}^+$ with $|H| > R$ we have*

$$\int_{\mathfrak{a}^*} h(\lambda)\Psi_\mu(\lambda, H)|c(\lambda)|^{-2}\, d\lambda = 0.$$

Proof. Using Remarks 9.3 and 9.4 we have for all $\lambda \in \mathfrak{a}^*$

$$\Psi_\mu(\lambda, H)|c(\lambda)|^{-2} = \sum_{s\in W} c(-s\lambda)^{-1}\Gamma_\mu(s\lambda)e^{(is\lambda-\rho-\mu)(H)}.$$

Thus by W-invariance of h it suffices to show

$$\int_{\mathfrak{a}^*} h(\lambda)c(-\lambda)^{-1}\Gamma_\mu(\lambda)e^{i\lambda(H)}\, d\lambda = 0 \tag{10.6}$$

for all $H \in \mathfrak{a}^+$ with $|H| > R$. To this end, we shall use Cauchy's residue theorem and shift the contour of integration. By Theorem 8.25 the possible poles of $\Gamma_\mu(\lambda)$ are located on the set of those $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that

$$2i\langle\lambda, \nu\rangle = \langle\nu, \nu\rangle \text{ for some } \nu \in \Lambda^+ \setminus \{0\}.$$

By the Gindikin-Karpelevic formula Theorem 9.1, $c(-\lambda)^{-1}$ is holomorphic outside the set of those $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that

$$\langle i\lambda, \alpha_0\rangle \in \tfrac{1}{2}\mathbb{Z}_{>0} \text{ for some } \alpha \in \Sigma_0^+.$$

In particular since $\Sigma_0 \subset \Lambda^+$, the function $c(-\lambda)^{-1}\Gamma_\mu(\lambda)$ is holomorphic outside the set of those $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that

$$\langle i\lambda, \nu\rangle > 0 \text{ for some } \nu \in \Lambda^+ \setminus \{0\}.$$

In particular if $\lambda_0 \in \mathfrak{a}_+^*$ then one may deform the contour $\mathfrak{a}^*$ to $\mathfrak{a}^* + i\lambda_0$ without running into poles of $c(-\lambda)^{-1}\Gamma_\mu(\lambda)$. In order to perform this contour shifting, we need to check that the integrand has sufficient decay as $|\Re(\lambda)| \to \infty$ with $\Im(\lambda) \in \mathfrak{a}_+^*$. By Theorem 8.25, in this region we have $|\Gamma_\mu(\lambda)| \ll_\mu 1$, and by Proposition 9.5 we have $|c(\lambda)|^{-1} \ll 1 + |\lambda|^{\frac{\dim N}{2}}$. Thus the integrand in 10.6 is

$$\ll_\mu |h(\lambda)|(1 + |\lambda|^{\frac{\dim N}{2}})e^{-\Im(\lambda)(H)}. \tag{10.7}$$

Thus choosing $M > 1 + \dim(\mathfrak{a}) + \frac{\dim N}{2}$ in (10.4), the contour shift is justified, hence

$$\int_{\mathfrak{a}^*} h(\lambda)c(-\lambda)^{-1}\Gamma_\mu(\lambda)e^{i\lambda(H)}\,d\lambda = \int_{\mathfrak{a}^*+i\lambda_0} h(\lambda)c(-\lambda)^{-1}\Gamma_\mu(\lambda)e^{i\lambda(H)}\,d\lambda.$$

Then using (10.4) and (10.7) we obtain

$$\int_{\mathfrak{a}^*+i\lambda_0} h(\lambda)c(-\lambda)^{-1}\Gamma_\mu(\lambda)e^{i\lambda(H)}\,d\lambda \ll_\mu e^{R|\lambda_0|-\lambda_0(H)}.$$

Now for H as above being fixed, choose $\lambda_0 = \langle tH, \cdot\rangle \in \mathfrak{a}^*_+$ for some $t > 0$. Then the right hand side in the above majoration equals $e^{t|H|(R-|H|)}$. Now letting $t \to \infty$ we deduce

$$\int_{\mathfrak{a}^*+i\lambda_0} h(\lambda)c(-\lambda)^{-1}\Gamma_\mu(\lambda)e^{i\lambda(H)}\,d\lambda = 0,$$

as needed. □

10.3. **Proof of the inversion theorem.** We now move on to the proof of Theorem 6.5. Given $g \in C_c^\infty(G)$ define

$$g^\#(x) = \int_K \int_K f(k_1 x k_2)\,dk_1 dk_2,$$

where as usual the Haar measure on K is normalised so that K has volume 1. Now consider the linear map $T : C_c^\infty(G) \to \mathbb{C}$ defined by

$$T(g) = \int_{\mathfrak{a}^*} \tilde{g^\#}(\lambda)|c(\lambda)|^{-2}\,d\lambda.$$

First we need to justify that the integral in the definition of T does make sense. Note that $g^\#$ obviously belongs to $C_c^\infty(G /\!\!/ K)$. Next given $f \in C_c^\infty(G /\!\!/ K)$ and using Corollary 5.9 we have for all $\lambda \in \mathfrak{a}^*$

$$\begin{aligned}
\tilde{f}(\lambda) &= \int_G f(x)\phi_{-\lambda}(x)\,dx \\
&= \int_G \int_K f(kx)e^{(\rho-i\lambda)A(kx)}\,dk\,dx = \int_G f(g)e^{(\rho-i\lambda)A(g)}\,dg \\
&= \int_A \int_N f(an)e^{(\rho-i\lambda)(\log a)}\,dnda \\
&= \int_A F_f(a)e^{-i\lambda(\log a)}\,da,
\end{aligned}$$

where we have used the notation of Theorem 5.15. Now it is clear that $F_f \in C_c^\infty(A)$. In fact the map $\mathfrak{a} \to \mathbb{C} : H \mapsto F_f(\exp H)$ belongs to $C_c^\infty(\mathfrak{a})$ and the last integral above is nothing but its usual Fourier transform. Thus by the usual Paley-Wiener

Theorem 1.2 it follows that $\tilde{f}$ is Paley-Wiener, and in particular using Proposition 9.5 the integral defining $T(g)$ does converge.

The first step of the proof of Theorem 6.5 is to show that T acts on the subspace $C_c^\infty(G \mathbin{/\!/} K)$ as the Dirac measure δ_{e_G}. Roughly, the idea is to insert the Harish-Chandra expansion of the spherical function in the definition of the spherical transform, and to use Proposition 10.6 after approximating the constant function 1 on $\mathfrak{a}^*$ by a Paley-Wiener function of arbitrarily small order. Namely, let F be a Paley-Wiener function of order 1 on $\mathfrak{a}^*_{\mathbb{C}}$, such that $F(0) = 1$ and such that F is radial (that is $F(\lambda)$ depends only on $\langle \lambda, \lambda \rangle$). The last condition ensures that F is in particular W-invariant. The existence of such a function F is provided by taking the Fourier transform of a suitable smooth, compactly supported, radial function on $\mathfrak{a}$. Then by the dominated convergence theorem and the Fubini theorem we have for all $g \in C_c^\infty(G)$

$$\begin{aligned} T(g) &= \lim_{\epsilon \to 0} \int_{\mathfrak{a}^*} F(\epsilon\lambda) \widetilde{g^\#}(\lambda) |c(\lambda)|^{-2}\, d\lambda \\ &= \lim_{\epsilon \to 0} \int_{\mathfrak{a}^*} F(\epsilon\lambda) \int_G g^\#(x) \phi_{-\lambda}(x)\, dx |c(\lambda)|^{-2}\, d\lambda \\ &= \lim_{\epsilon \to 0} \int_G g^\#(x) h_\epsilon(x)\, dx, \end{aligned}$$

where

$$h_\epsilon(x) = \int_{\mathfrak{a}^*} F(\epsilon\lambda) \phi_{-\lambda}(x) |c(\lambda)|^{-2}\, d\lambda.$$

Now inserting Harish-Chandra's expansion (10.5) and using (8.13) to justify the termwise integration, we have for $x \in K \exp(H) K$ with $H \in \mathfrak{a}_+$

$$h_\epsilon(x) = \sum_{\mu \in \Lambda^+} \int_{\mathfrak{a}^*} F(\epsilon\lambda) \Psi_\mu(\lambda, H) |c(\lambda)|^{-2}\, d\lambda.$$

But the function $\lambda \mapsto F(\epsilon\lambda)$ is Paley-Wiener of order ϵ and thus by Proposition 10.6 and the Cartan decomposition Theorem 4.36, the function h_ϵ is supported on $\mathscr{B}_\epsilon = K\{\exp(H) : H \in \overline{\mathfrak{a}_+} : |H| \leq \epsilon\}K$. In particular observing that if $\mathrm{Supp}(g) \cap K = \varnothing$ then $\mathrm{Supp}(g^\#) \cap K = \varnothing$ too, we have established the following important result.

Fact 2. *If* $\mathrm{Supp}(g) \cap K = \varnothing$ *then* $T(g) = 0$.

Now we extend the operator T to all of $C_c(G)$. Since $C_c^\infty(G)$ is dense in $C_c(G)$ endowed with the uniformity norm $\|\cdot\|_\infty$ (which may be shown by using local charts to "lift" the corresponding statement from $\mathbb{R}^n$), it suffices to show that there exists a constant c such that

$$|T(g)| \le c\|g\|_\infty \tag{10.8}$$

for all $g \in C_c^\infty(G)$. But using the support condition for h_ϵ and the obvious majoration $\|g^\#\|_\infty \le \|g\|_\infty$ we have

$$|T(g)| \le \limsup_{\epsilon\to 0} \int_G |g^\#(x)h_\epsilon(x)|\,dx \le \limsup_{\epsilon\to 0} \|h_\epsilon\|_\infty \|g\|_\infty \int_G \mathbb{1}_{\mathscr{B}_\epsilon}(x)\,dx. \tag{10.9}$$

So we need to estimate on the one hand $\|h_\epsilon\|_\infty$ and on the other hand the volume of $\mathscr{B}_\epsilon$. We start with the estimation of $\|h_\epsilon\|_\infty$. Let $\epsilon < 1$. Firstly by continuity of the map $g \mapsto e^{(\rho - i\lambda)(g)}$ and by compactness we have for all $\lambda \in \mathfrak{a}^*$ and for $x \in \mathscr{B}_\epsilon$

$$|\phi_{-\lambda}(x)| \le \max_{x\in\mathscr{B}_\epsilon} e^{\rho(A(x))} \le C,$$

where $C = \max_{x\in\mathscr{B}_1} e^{\rho(A(x))}$ say. Thus using Proposition 9.5 we have

$$\begin{aligned}\|h_\epsilon\|_\infty &\le C\int_{\mathfrak{a}^*} |F(\epsilon\lambda)c(\lambda)|^{-2}\,d\lambda = C\epsilon^{-\dim\mathfrak{a}}\int_{\mathfrak{a}^*} |F(\lambda)c(\epsilon^{-1}\lambda)|^{-2}\,d\lambda \\ &\le C\epsilon^{-\dim\mathfrak{a}}\int_{\mathfrak{a}^*} |F(\lambda)|\left(C_1 + C_2|\epsilon^{-1}\lambda|^{\frac{\dim N}{2}}\right)^2 d\lambda,\end{aligned}$$

and thus

$$\|h_\epsilon\|_\infty \ll \epsilon^{-\dim NA}. \tag{10.10}$$

To calculate the volume of $\mathscr{B}_\epsilon$ we shall use the following.

Lemma 10.7. *Let $\mathscr{M}$ and $\mathscr{N}$ be oriented manifolds endowed with n-forms $\omega^{\mathscr{M}}$ and $\omega^{\mathscr{N}}$ respectively. Let Φ be an orientation-preserving diffeomorphism from $\mathscr{M}$ onto $\mathscr{N}$. Let $H \subset \mathscr{M}$ be compact. Then there exists a constant C_H such that for all measurable subset $B \subset H$ we have*

$$\int_{\mathscr{M}} \mathbb{1}_B\omega^{\mathscr{M}} \le C_H \int_{\mathscr{N}} \mathbb{1}_{\Phi(B)}\omega^{\mathscr{N}}$$

Proof. We use the change of variable formula (5.3) and given $p \in K$ and $\ell \in \mathrm{Hom}(\mathbb{R}^n, T_p\mathscr{M})$ we write

$$(\Phi^*\omega^{\mathscr{N}})_p(\ell) = \omega_p^{\mathscr{M}}(\ell)\det(\Omega^{\mathscr{N}}_{\phi(p)} \circ d_p\Phi \circ (\Omega_p^{\mathscr{M}})^{-1}).$$

By compactness of H the continuous map

$$p \mapsto |\det(\Omega^{\mathscr{N}}_{\phi(p)} \circ d_p\Phi \circ (\Omega_p^{\mathscr{M}})^{-1})|$$

is bounded on H. □

Now we apply the previous lemma to the diffeomorphism $\Phi : G \to \mathfrak{p} \times K$ provided by Theorem 4.25 with $H = \mathscr{B}_1$ say and $B = \mathscr{B}_\epsilon$. We claim that in this case $\Phi(B) = \{X \in \mathfrak{p} : |X| \leq \epsilon\} \times K$. Indeed, assume

$$k_1 \exp(H) k_2 = \exp(X)$$

where $k_1, k_2 \in K$, $H \in \mathfrak{a}$ and $X \in \mathfrak{p}$. Using that $k_1 \exp(H) k_2 = \exp(\mathrm{Ad}_G(k_1)H) k_1^{-1} k_2$ and $\mathrm{Ad}_G(K)\mathfrak{a} = \mathfrak{p}$ by Lemma 4.27, we must have $k_1 = k_2$ and $X = \mathrm{Ad}_G(k)H$. But then by (4.3) we have $|X| = |H|$. Thus we get

$$\int_G \mathbb{1}_{\mathscr{B}_\epsilon}(x)\, dx \ll \epsilon^{\dim(\mathfrak{p})}. \tag{10.11}$$

Since $\dim(\mathfrak{p}) = \dim(NA)$, combining (10.9),(10.10) and (10.11) prove ((10.8)).

Thus T extends to a continuous linear map $C_c(G) \to \mathbb{C}$. By the Riesz–Markov–Kakutani representation theorem, there exists a (complex-valued) Borel measure μ on G such that $T(f) = \int_G f\, d\mu$ for all $f \in C_c(G)$. Fact 2 then implies that μ is supported on K, and in particular for all $f \in C_c^\infty(G /\!\!/ K)$ we have

$$T(f) = f(e_G) \int_K d\mu.$$

In addition, we must have $\int_K d\mu > 0$ because it is possible to find $g \in C_c^\infty(G /\!\!/ K)$ such that $\tilde{g}(\lambda) > 0$ for all $\lambda \in \mathfrak{a}^*$ (to see this, use Remark 10.4).

Thus we have obtained

$$\int_{\mathfrak{a}^*} \tilde{g}(\lambda) |c(\lambda)|^{-2}\, d\lambda = f(e_G). \tag{10.12}$$

for all $f \in C_c^\infty(G /\!\!/ K)$, which is the statement of the inversion theorem for $g = e_G$. Now let $g \in G$ be arbitrary and consider the function $h \in C_c^\infty(G /\!\!/ K)$ defined by

$$h(x) = \int_K f(gkx)\, dk.$$

Then using Lemmas 10.1 and 10.3 we have

$$\begin{aligned} \tilde{h}(\lambda) &= \int_K \int_G f(gkx) \phi_{-\lambda}(x)\, dx dk \\ &= \int_G f(x) \phi_{-\lambda}(g^{-1}x)\, dx = \int_G f(x) \int_K \phi_{-\lambda}(g^{-1}kx)\, dx \\ &= \phi_{-\lambda}(g^{-1}) \tilde{f}(\lambda) = \phi_\lambda(g) \tilde{f}(\lambda). \end{aligned}$$

Applying (10.12) to h we obtain the inversion formula. To deduce the second statement of Theorem 6.5 we apply (10.12) to the function

$$x \mapsto \int_G f(xg)\overline{f(g)}\, dg$$

and we use Corollary 10.4.

10.4. **Proof of the Paley-Wiener theorem.** We have already justified above the fact that the spherical transform of an element of $C_c^\infty(G /\!\!/ K)$ is Paley-Wiener. The W-invariance follows directly from the W-invariance of the spherical function, and the injectivity is a consequence of the inversion formula. Thus we need to show the surjectivity. So let h be a W-invariant Paley-Wiener function on $\mathfrak{a}_{\mathbb{C}}^*$ of order R, say. Define

$$f(x) = \int_{\mathfrak{a}^*} h(\lambda)\phi_\lambda(x)|c(\lambda)|^{-2}\, d\lambda.$$

Note that the integral is well defined by Proposition 9.5 and the Paley-Wiener assumption. Now for any $D \in D(G)$ the function $(D\phi_\lambda)(x)$ is bounded by a polynomial in λ, uniformly for x varying in a fixed compact set. Thus by differentiation under the integral sign, we deduce that $f \in C^\infty(G /\!\!/ K)$. Now similarly as in the proof of Theorem 6.5 we insert the Harish-Chandra (10.5) we use (8.13). Thus the localisation property Proposition 10.6 implies that $f \in C_c^\infty(G /\!\!/ K)$. It remains to prove that $h = \tilde{f}$. By the inversion theorem we have

$$\int_{\mathfrak{a}^*} (\tilde{f}(\lambda) - h(\lambda))\phi_\lambda(x)|c(\lambda)|^{-2}\, d\lambda = 0.$$

Integrating this against an arbitrary $g \in C_c^\infty(G)$ we deduce

$$\int_{\mathfrak{a}^*} (\tilde{f}(\lambda) - h(\lambda))\tilde{g}(\lambda)|c(\lambda)|^{-2}\, d\lambda = 0 \tag{10.13}$$

for all $g \in C_c^\infty(G)$.

Lemma 10.8. *Let $\mathscr{A}$ be the image of $C_c^\infty(G /\!\!/ K)$ under the map $f \mapsto \tilde{f}_{|\mathfrak{a}^*}$. Then $\mathscr{A}$ is an algebra that is closed under complex conjugation, it separates the points on $\mathfrak{a}^*/W$, and vanishes at infinity.*

Proof. The first statement is a reformulation of Corollary 10.4. To prove that $\mathscr{A}$ separates the points on $\mathfrak{a}^*/W$, let $\lambda, \mu \in \mathfrak{a}^*$ and assume that $\tilde{h}(\lambda) = \tilde{h}(\mu)$ for all $h \in C_c^\infty(G /\!\!/ K)$, that is

$$\int_G h(g)(\phi_\lambda(g) - \phi_\mu(g))\, dg = 0$$

for all $h \in C_c^\infty(G /\!\!/ K)$. Then $\phi_\lambda(g) = \phi_\mu(g)$ for all $g \in G$ and hence by Harish-Chandra's Theorem 6.4 we have $\lambda = s\mu$ for some $s \in W$, as desired. Finally functions in $\mathscr{A}$ vanish at infinity because they are Paley-Wiener. $\square$

Thus by the Stone-Weierstraß theorem, $\mathscr{A}$ is dense (with respect to the uniformity norm) in the space of all continuous functions on $\mathfrak{a}^*/W$ that vanish at infinity. Thus (10.13) implies that $\tilde{f}$ and h coincide on $\mathfrak{a}^*$, and by holomorphy $\tilde{f} = h$ on $\mathfrak{a}^*_{\mathbb{C}}$, finishing the proof and the course.

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